24/03/2014 - P1 DIN2 - Prof. Thiago Ritto

1) (2,0) Obtenha o momento de inércia I_{xx} e o produto de inércia I_{xz} em relação ao ponto O da estrutura mostrada na Fig. 1, usando a base solidária à própria estrutura. Duas placas quadradas (m_p, L) e uma barra $(m_b, 2L)$, todas soldadas. O tensor de inércia calculado nessa base é o mesmo do calculado em uma base fixa no referencial inercial? Justifique.

2) (3,5) Considere o sistema mostrado na Fig. 2, que é uma simplificação de uma broca tricônica usada em uma coluna de perfuração de petróleo. Um cone (m_d, D) que gira em torno do seu eixo de simetria com velocidade angular $\omega\mathbf{d}_2$ e está conectado a uma mesa giratória que gira $\Omega\mathbf{a}_2$, conforme mostra a figura. Além disso, a mesa se move com velocidade linear $v\mathbf{a}_2$. Calcule: (a) a quantidade de movimento angular do cone em relação ao ponto O, e (b) a energia cinética do cone. O tensor de inércia do cone na base $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ em relação ao seu centro de massa (CM) é diagonal com componentes conhecidas I_x, I_y e I_z. A base $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ está fixa no referencial inercial, a base $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ e $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ são solidárias à mesa, que gira $\Omega\mathbf{a}_2$, e a base $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ é solidária ao cone.

3) (3,0) Um fio inextensível é enrolado em um ioio (m, I^c) e conectado a uma mola de rigidez k, conforme mostrado na Fig. 3. Quando $x = 0$, a mola está em equilibrio. Pede-se (a) o diagrama de corpo livre do ioio, (b) as energias cinética e potencial do sistema. Obtenha as equações de movimento do ioio (Newton OU Lagrange) considerando (c) que o ioio rola sem deslizar (d) que o ioio rola deslizando.

4) (1,5) Um impulso $\mathbf{I} = F\Delta t\mathbf{a}_1$ é aplicado no sistema (inicialmente em repouso) mostrado na Fig. 4, formado por duas massas pontuais e uma barra de massa desprezível. Calcule (a) a posição do centro de massa do sistema, (b) a vetor velocidade do centro de massa do sistema após o impacto, (b) a vetor velocidade angular do sistema após o impacto. (c) Se o impulso fosse aplicado na massa de baixo, a velocidade angular aumentaria, diminuiria, ou seria a mesma? Justifique.

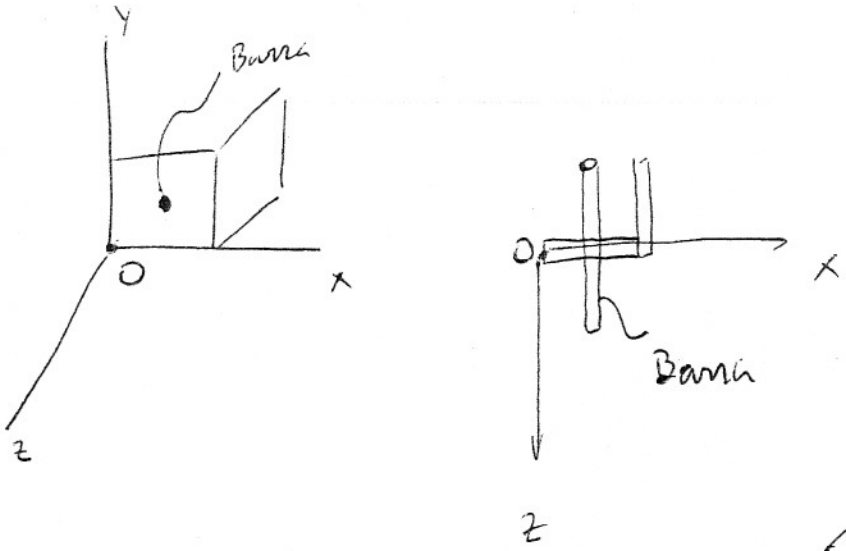


FIG. 1

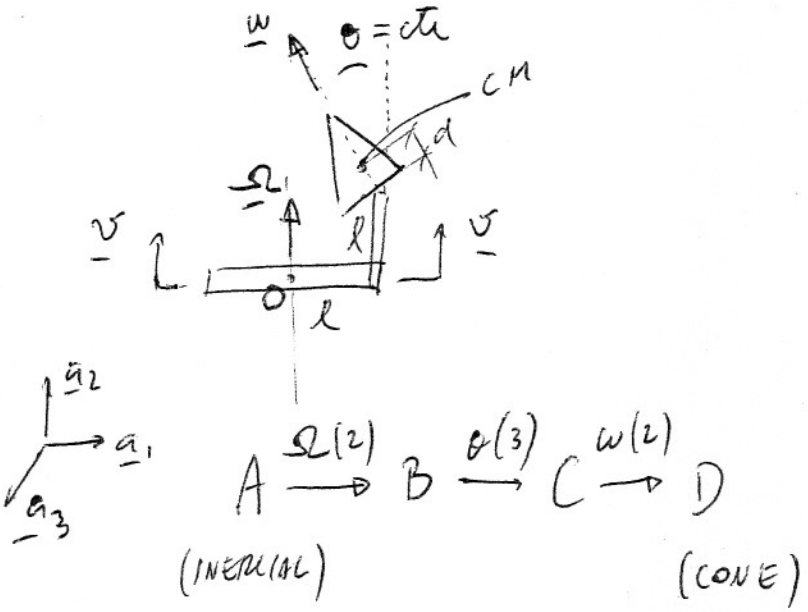


FIG. 2

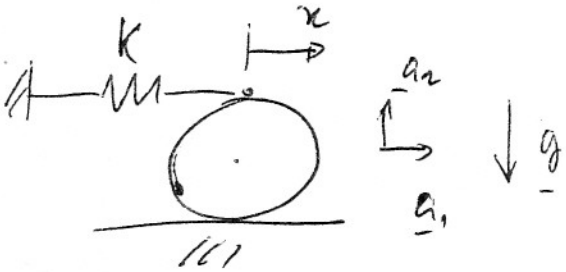


FIG. 3

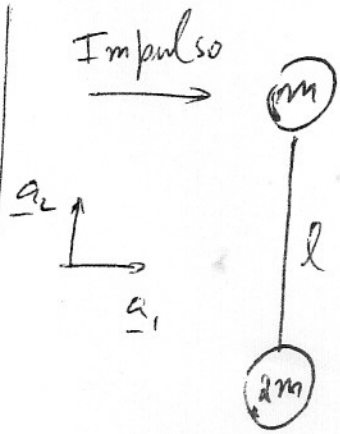
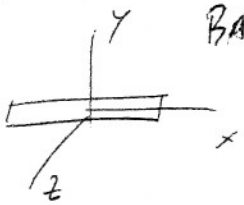


FIG. 4

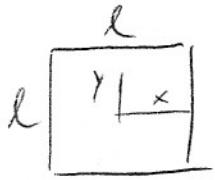
TABELA



BARRA

$I_{yy} = I_{zz} = \frac{1}{12} m l^2$

$I_{xx} = I_{xy} = I_{xz} = I_{yz} = 0$



PLACA

$I_{xx} = I_{yy} = m l^2/12$

$I_{zz} = m l^2/6$

$I_{xy} = I_{xz} = I_{yz} = 0$

P1. 2014.1

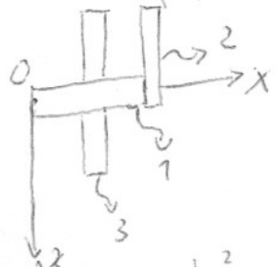
1) $[I_{xx}^{OS}] = [I_{xx}^{01}] + [I_{xx}^{02}] + [I_{xx}^{03}]$; $[I_{xz}^{OS}] = [I_{xz}^{01}] + [I_{xz}^{02}] + [I_{xz}^{03}]$

$$I_{xx}^{01} = I_{xx}^{CM1} + m_p(d_y^2 + d_z^2) = \frac{m_p L^2}{12} + m_p\left(\frac{L^2}{4} + 0\right) = \frac{1}{3} m_p L^2$$

$$I_{xx}^{02} = I_{xx}^{CM2} + m_p(d_y^2 + d_z^2) = \frac{m_p L^2}{6} + m_p\left(\frac{L^2}{4} + \frac{L^2}{4}\right) = \frac{2}{3} m_p L^2$$

$$I_{xx}^{03} = I_{xx}^{CM3} + m_b(d_y^2 + d_z^2) = \frac{m_b}{12}(4L^2) + m_b\left(\frac{L^2}{4} + 0\right) = \frac{7}{12} m_b L^2$$

$$[I_{xx}^{OS}] = m_p L^2 + \frac{7}{12} m_b L^2$$



$$I_{xz}^{01} = \cancel{I_{xz}^{CM1}}^{\,0} + m_p \cancel{d_x d_z}^{\,0} = 0$$

$$I_{xz}^{02} = \cancel{I_{xz}^{CM2}}^{\,0} + m_p d_x d_z = m_p(L)(-L/2) = -\frac{m_p L^2}{2}$$

$$I_{xz}^{03} = \cancel{I_{xz}^{CM3}}^{\,0} + m_b \cancel{d_x d_z}^{\,0} = 0$$

$$I_{xz}^{OS} = -\frac{m_p L^2}{2}$$

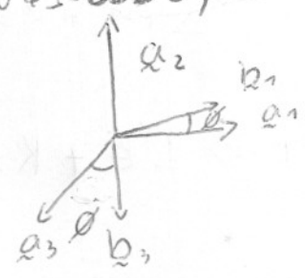
Não, já que poderá haver um giro ou movimento de translação do sistema em relação ao ponto O, o que altera o tensor de inércia. (no ref. inercial, aparecerão senos e cossenos referentes aos giros)

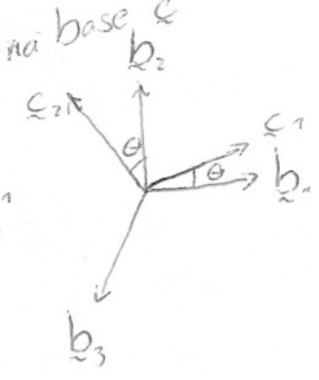
2) a) $\underset{\sim}{H}^{C/O} = \underset{\sim}{H}^{C/CM} + \underset{\sim}{r}^{CM/O} \times \underset{\sim}{G}^{C}$

Como o corpo apresenta simetria em torno do eixo onde ocorre o último giro, pode-se fazer todos os cálculos na base C.

$$\underset{\sim}{H}^{C/CM} = [I^{CM}]\,{}^{R}\underset{\sim}{\omega}^{D}$$

${}^{R}\underset{\sim}{\omega}^{D} = (\Omega\, \underset{\sim}{b}_2 + \omega\, \underset{\sim}{c}_2) = (-\Omega \operatorname{sen}\theta, \omega + \Omega\cos\theta, 0)$ na base $\underset{\sim}{c}$





$$H^{C/CM} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \begin{bmatrix} -\Omega \operatorname{sen}\theta \\ \omega + \Omega\cos\theta \\ 0 \end{bmatrix}$$

$$= I_y(\omega + \Omega\cos\theta)\,\underset{\sim}{c}_2 + I_x \Omega \operatorname{sen}\theta\, \underset{\sim}{c}_1$$

$$\underset{\sim}{p}^{CM/O} = L\,\underset{\sim}{a}_1 + L\,\underset{\sim}{a}_2 + d\,\underset{\sim}{c}_2 = L\cos\phi\,\underset{\sim}{b}_1 + L\,\mathrm{sen}\,\phi\,\underset{\sim}{b}_3 + L\,\underset{\sim}{b}_2 + d\,\underset{\sim}{c}_2$$

$$= \left[L(\cos\phi\cos\theta + \mathrm{sen}\,\theta)\,\underset{\sim}{c}_1 + \left[d + L(\cos\theta - \cos\phi\,\mathrm{sen}\,\theta)\right]\underset{\sim}{c}_2 + L\,\mathrm{sen}\,\phi\,\underset{\sim}{c}_3 \right]$$

$$\underset{\sim}{G}^{C} = m_C\,{}^{R}\underset{\sim}{v}^{CM}$$

$$\begin{cases} {}^{R}\underset{\sim}{v}^{O} = v\,\underset{\sim}{b}_2 = v\cos\theta\,\underset{\sim}{c}_2 + v\,\mathrm{sen}\,\theta\,\underset{\sim}{c}_1 \\ {}^{R}\underset{\sim}{\omega}^{C} = \Omega\,\underset{\sim}{b}_2 = \Omega\cos\theta\,\underset{\sim}{c}_2 + \Omega\,\mathrm{sen}\,\theta\,\underset{\sim}{c}_1 \end{cases}$$

$${}^{R}\underset{\sim}{v}^{CM} = {}^{R}\underset{\sim}{v}^{O} + \cancel{{}^{C}\underset{\sim}{v}^{CM}}^{\,0} + {}^{R}\underset{\sim}{\omega}^{C} \times \underset{\sim}{p}^{CM/O}$$

$${}^{R}\underset{\sim}{\omega}^{C} \times \underset{\sim}{p}^{CM/O} = \Omega\,\mathrm{sen}\,\theta\left[d + L(\cos\theta - \cos\phi\,\mathrm{sen}\,\theta)\right]\underset{\sim}{c}_3 - \Omega\,\mathrm{sen}\,\theta\,L\,\mathrm{sen}\,\phi\,\underset{\sim}{c}_2 -$$

$$- \Omega\cos\theta\,L(\cos\phi\cos\theta + \mathrm{sen}\,\theta)\,\underset{\sim}{c}_3 + \Omega\cos\theta\,L\,\mathrm{sen}\,\phi\,\underset{\sim}{c}_1$$

$${}^{R}\underset{\sim}{v}^{CM} = \Big[\left(v\,\mathrm{sen}\,\theta + \Omega\cos\theta\,L\,\mathrm{sen}\,\phi\right)\underset{\sim}{c}_1 + \left(v\cos\theta - \Omega\,\mathrm{sen}\,\theta\,L\,\mathrm{sen}\,\phi\right)\underset{\sim}{c}_2 +$$

$$+ \left\{ \Omega\,\mathrm{sen}\,\theta\left[d + L(\cos\theta - \cos\phi\,\mathrm{sen}\,\theta)\right] - \Omega\cos\theta\,L(\cos\phi\cos\theta + \mathrm{sen}\,\theta) \right\}\underset{\sim}{c}_3 \Big]$$

$$\underset{\sim}{p}^{CM/O} \times m_C\,{}^{R}\underset{\sim}{v}^{CM} = m_C \Big\{ L(\cos\phi\cos\theta + \mathrm{sen}\,\theta)\left(v\cos\theta - \Omega\,\mathrm{sen}\,\theta\,L\,\mathrm{sen}\,\phi\right)\underset{\sim}{c}_3 +$$

$$+ L(\cos\phi\cos\theta + \mathrm{sen}\,\theta)\Big[\Omega\cos\theta\,L(\cos\phi\cos\theta + \mathrm{sen}\,\theta) - \Omega\,\mathrm{sen}\,\theta\left[d + L(\cos\theta -$$

$$- \cos\phi\,\mathrm{sen}\,\theta)\right]\Big]\underset{\sim}{c}_2 + \left[L(\cos\phi\,\mathrm{sen}\,\theta - \cos\theta) - d\right]\left[v\,\mathrm{sen}\,\theta + \Omega L\,\mathrm{sen}\,\phi\cos\theta\right]\underset{\sim}{c}_3 +$$

$$+ \left[d + L(\cos\theta - \cos\phi\,\mathrm{sen}\,\theta)\right]\Big[\Omega\,\mathrm{sen}\,\theta\left(d + L(\cos\theta - \cos\phi\,\mathrm{sen}\,\theta)\right) - \Omega\cos\theta\,L(\cos\phi\cos\theta +$$

$$+ \mathrm{sen}\,\theta)\Big]\underset{\sim}{c}_1 + L\,\mathrm{sen}\,\phi\left(v\,\mathrm{sen}\,\theta + \Omega\cos\theta\,L\,\mathrm{sen}\,\phi\right)\underset{\sim}{c}_2 + L\,\mathrm{sen}\,\phi\left(\Omega L\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi -$$

$$- v\cos\theta\right)\underset{\sim}{c}_1 \Big\}$$

$$H^{C/O} = \begin{bmatrix} I_x\Omega\,\mathrm{sen}\,\theta + m_C\Big\{\left[d + L(\cos\theta - \cos\phi\,\mathrm{sen}\,\theta)\right]\Big[\Omega\,\mathrm{sen}\,\theta\big(d + L(\cos\theta - \\ - \cos\phi\,\mathrm{sen}\,\theta)\big) - \Omega\cos\theta\,L(\cos\phi\cos\theta + \mathrm{sen}\,\theta) + L\,\mathrm{sen}\,\phi(\Omega L\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi - \\ - v\cos\theta)\Big\} \\ I_y(\omega + \Omega\cos\theta) + m_C\Big\{L(\cos\phi\cos\theta + \mathrm{sen}\,\theta)\Big[\Omega\cos\theta\,L(\cos\phi\cos\theta + \\ + \mathrm{sen}\,\theta) - \Omega\,\mathrm{sen}\,\theta\left[d + L(\cos\theta - \cos\phi\,\mathrm{sen}\,\theta)\right]\Big] + L\,\mathrm{sen}\,\phi(v\,\mathrm{sen}\,\theta + \\ \Omega\cos\theta\,L\,\mathrm{sen}\,\phi)\Big\} \\ m_C\Big\{L(\cos\phi\cos\theta + \mathrm{sen}\,\theta)\left(v\cos\theta - \Omega\,\mathrm{sen}\,\theta\,L\,\mathrm{sen}\,\phi\right) + \left[L(\cos\phi\,\mathrm{sen}\,\theta - \\ \cos\theta) - d\right]\left[v\,\mathrm{sen}\,\theta + \Omega L\,\mathrm{sen}\,\phi\cos\theta \end{bmatrix} \quad \text{na base } \underset{\sim}{c}$$

2) b) $K = \frac{1}{2}\underset{\sim}{\omega}^T [I^{CM}] \underset{\sim}{\omega} + \frac{m_c}{2} \overset{R}{\underset{\sim}{v}}{}^{CM\,T} \cdot \overset{R}{\underset{\sim}{v}}{}^{CM}$

$$= \frac{1}{2}\left(-\Omega \operatorname{sen}\theta,\ \omega + \Omega\cos\theta,\ 0\right)\begin{pmatrix} -I_x \Omega \operatorname{sen}\theta \\ I_y(\omega + \Omega\cos\theta) \\ 0 \end{pmatrix} + \frac{m_c}{2} \overset{R}{\underset{\sim}{v}}{}^{CM\,T} \cdot \overset{R}{\underset{\sim}{v}}{}^{CM}$$

$$= \frac{1}{2}\left(I_x \Omega^2 \operatorname{sen}^2\theta + I_y(\omega + \Omega\cos\theta)^2 + \frac{m_c}{2} \overset{R}{v}{}^{CM\,T} \cdot \overset{R}{v}{}^{CM}\right.$$

$$= \frac{1}{2}\left(I_x \Omega^2 \operatorname{sen}^2\theta + I_y(\omega + \Omega\cos\theta)^2 + \frac{m_c}{2}\left[\left(v \operatorname{sen}\theta + \Omega L \cos\theta \operatorname{sen}\phi\right)^2 + \right.\right.$$

$$+ \left(v\cos\theta - \Omega L \operatorname{sen}\theta \operatorname{sen}\phi\right)^2 + \left\{\Omega \operatorname{sen}\theta\left[d + L\left(\cos\theta - \cos\phi \operatorname{sen}\theta\right)\right] - \right.$$

$$\left.\left. - \Omega\cos L\left(\operatorname{sen}\theta + \cos\theta\cos\phi\right)\right\}^2$$

3) a)

b) $K = \frac{1}{2} I^c \dot{\theta}^2 + \frac{1}{2} m \dot{x}^2$ $\qquad \delta = x + \theta r$

$$\phi = \frac{K\delta^2}{2} = \frac{K}{2}\left(x^2 + 2\theta x r + \theta^2 r^2\right)$$

c) Rola sem deslizar $\Leftrightarrow W^{FNC} = 0$ e $x = \theta r \Rightarrow \dot{x} = \dot{\theta} r$

$$L = K - \phi = \frac{1}{2} I^c \dot{\theta}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{K}{2} 4\theta^2 r^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \left(I^c + m r^2\right)\ddot{\theta} + 4K\theta r^2 = 0$$

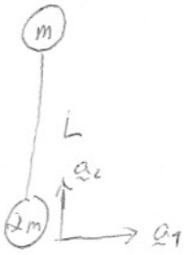
d) Rola deslizando $\Leftrightarrow$ x e θ independentes.

$$L = \frac{1}{2} I^c \dot{\theta}^2 + \frac{1}{2} m \dot{x}^2 - \frac{K}{2}\left(x^2 + 2\theta x r + \theta^2 r^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = F_{at} r \Rightarrow I^c \ddot{\theta} + K x r + K\theta r^2 - \mu_D m g r = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = -F_{at} \Rightarrow m\ddot{x} + Kx + K\theta r + \mu_D m g = 0$$

4) a)



$$\underset{\sim}{x}^{CM} = \frac{(m \times 0 + 2m \times 0)}{3m} = 0\,\underset{\sim}{a}_1$$

$$\underset{\sim}{x}^{c} = \frac{(mL\,\underset{\sim}{a}_2 + 2m \times 0)}{3m} = \frac{1}{3} L\,\underset{\sim}{a}_2$$

b) $\underset{\sim}{I} = \Delta \underset{\sim}{G} = \underset{\sim}{G}^{D} - \cancel{\underset{\sim}{G}^{A}}^{\,0}$

$F \Delta t\, \underset{\sim}{a}_1 = 3m\, {}^{R}\underset{\sim}{v}^{CM}$

${}^{R}\underset{\sim}{v}^{CM} = \dfrac{F \Delta t}{3m}\, \underset{\sim}{a}_1$

c) Na massa de cima:

$\underset{\sim}{I}^{ang} = \Delta \underset{\sim}{H} = \underset{\sim}{H}^{D} - \cancel{\underset{\sim}{H}^{A}}^{\,0}$

$\underbrace{\frac{2}{3} F \Delta t L = I^{CM} \underset{\sim}{\omega}}_{E_m \cdot \underset{\sim}{a}_3}$

$\underset{\sim}{\omega} = -\dfrac{2 F \Delta t L}{3 I^{CM}}\, \underset{\sim}{a}_3$

Na massa de baixo

$\underset{\sim}{I}^{ang} = \Delta \underset{\sim}{H} = \underset{\sim}{H}^{D} - \cancel{\underset{\sim}{H}^{A}}^{\,0}$

$\underbrace{\frac{1}{3} F \Delta t L = I^{CM} \underset{\sim}{\omega}}_{E_m + \underset{\sim}{a}_3}$

$\underset{\sim}{\omega} = +\dfrac{F \Delta t L}{3 I^{CM}}\, \underset{\sim}{a}_3$

Em módulo, ela seria menor, devido à menor distância em relação ao c.m..