DSL - Dinâmica de Sistemas Lineares (e CONTROLE)

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Referências:

Ogata, Engenharia de Controle Moderno, 5a edição, 2011

Boyce e DiPrima, Elementary Differential Equations and Boundary Value Problems, 2001

Table A–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$

13	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s + a)}$
15	$\frac{1}{b - a}(e^{-at} - e^{-bt})$	$\frac{1}{(s + a)(s + b)}$
16	$\frac{1}{b - a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s + a)(s + b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a - b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s + a)(s + b)}$

(*continues on next page*)

Table A–1 (*continued*)

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s + a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s + a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1 - \zeta^2}\, t \quad (0 < \zeta < 1)$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t - \phi\right)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \phi\right)$ $\phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0 < \zeta < 1, \quad 0 < \phi < \pi/2)$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2} (\cos \omega_1 t - \cos \omega_2 t) \qquad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega} (\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

Table A–2 Properties of Laplace Transforms

1	$\mathcal{L}[Af(t)] = AF(s)$
2	$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathcal{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathcal{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathcal{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathcal{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\,dt\right]_{t=0\pm}$
7	$\mathcal{L}_\pm\left[\int\cdots\int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n}\frac{1}{s^{n-k+1}}\left[\int\cdots\int f(t)(dt)^k\right]_{t=0\pm}$
8	$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
9	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \quad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
10	$\mathcal{L}[e^{-\alpha t}f(t)] = F(s+\alpha)$
11	$\mathcal{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \quad \alpha \ge 0$
12	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
13	$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
14	$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \quad (n = 1, 2, 3, \dots)$
15	$\mathcal{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds \quad \text{if } \lim_{t\to 0}\frac{1}{t}f(t) \text{ exists}$

16	$\mathcal{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$
17	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$
18	$\mathcal{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp$

Finally, we present two frequently used theorems, together with Laplace transforms of the pulse function and impulse function.

Initial value theorem	$f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$
Final value theorem	$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$
Pulse function $f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t-t_0)$	$\mathcal{L}[f(t)] = \frac{A}{t_0 s} - \frac{A}{t_0 s}e^{-st_0}$

Pulse function $f(t) = \frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)$	$\mathcal{L}[f(t)] = \frac{A}{t_0 s} - \frac{A}{t_0 s} e^{-st_0}$
Impulse function $g(t) = \lim_{t_0 \to 0} \frac{A}{t_0}, \quad \text{for } 0 < t < t_0$ $= 0, \quad \text{for } t < 0, t_0 < t$	$\mathcal{L}[g(t)] = \lim_{t_0 \to 0} \left[\frac{A}{t_0 s} (1 - e^{-st_0}) \right]$ $= \lim_{t_0 \to 0} \frac{\frac{d}{dt_0}[A(1 - e^{-st_0})]}{\frac{d}{dt_0}(t_0 s)}$ $= \frac{As}{s} = A$

Convolution Integral

Theorem 6.6.1 If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \qquad s > a, \tag{1}$$

where

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{2}$$

The function h is known as the convolution of f and g; the integrals in Eq. (2) are known as convolution integrals.

The equality of the two integrals in Eq. (2) follows by making the change of variable $t - \tau = \xi$ in the first integral. Before giving the proof of this theorem let us make some observations about the convolution integral. According to this theorem, the transform of the convolution of two functions, rather than the transform of their ordinary product, is given by the product of the separate transforms. It is conventional to emphasize that the convolution integral can be thought of as a "generalized product" by writing

$$h(t) = (f * g)(t). \tag{3}$$

In particular, the notation $(f * g)(t)$ serves to indicate the first integral appearing in Eq. (2).

The convolution $f * g$ has many of the properties of ordinary multiplication. For example, it is relatively simple to show that

$$f * g = g * f \qquad \text{(commutative law)} \tag{4}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)} \tag{5}$$

$$(f * g) * h = f * (g * h) \qquad \text{(associative law)} \tag{6}$$

$$f * 0 = 0 * f = 0. \tag{7}$$

The proofs of these properties are left to the reader. However, there are other properties of ordinary multiplication that the convolution integral does not have. For example, it is not true in general that $f * 1$ is equal to f. To see this, note that

$$(f * 1)(t) = \int_0^t f(t - \tau) \cdot 1 \, d\tau = \int_0^t f(t - \tau) \, d\tau.$$

If, for example, $f(t) = \cos t$, then

$$\begin{aligned} (f * 1)(t) &= \int_0^t \cos(t - \tau) \, d\tau = -\sin(t - \tau)\Big|_{\tau=0}^{\tau=t} \\ &= -\sin 0 + \sin t \\ &= \sin t. \end{aligned}$$

Clearly, $(f * 1)(t) \neq f(t)$. Similarly, it may not be true that $f * f$ is nonnegative. See Problem 3 for an example.

Convolution integrals arise in various applications in which the behavior of the system at time t depends not only on its state at time t, but on its past history as well. Systems of this kind are sometimes called hereditary systems and occur in such diverse fields as neutron transport, viscoelasticity, and population dynamics.

Turning now to the proof of Theorem 6.6.1, we note first that if

$$F(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi$$

and

$$G(s) = \int_0^\infty e^{-s\eta} g(\eta)\, d\eta,$$

then

$$F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \int_0^\infty e^{-s\eta} g(\eta)\, d\eta. \tag{8}$$

Since the integrand of the first integral does not depend on the integration variable of the second, we can write $F(s)G(s)$ as an iterated integral,

$$F(s)G(s) = \int_0^\infty g(\eta)\, d\eta \int_0^\infty e^{-s(\xi+\eta)} f(\xi)\, d\xi. \tag{9}$$

This expression can be put into a more convenient form by introducing new variables of integration. First let $\xi = t - \eta$, for fixed η. Then the integral with respect to ξ in Eq. (9) is transformed into one with respect to t; hence

$$F(s)G(s) = \int_0^\infty g(\eta)\, d\eta \int_\eta^\infty e^{-st} f(t-\eta)\, dt. \tag{10}$$

Next let $\eta = \tau$; then Eq. (10) becomes

$$F(s)G(s) = \int_0^\infty g(\tau)\, d\tau \int_\tau^\infty e^{-st} f(t-\tau)\, dt. \tag{11}$$

The integral on the right side of Eq. (11) is carried out over the shaded wedge-shaped region extending to infinity in the $t\tau$-plane shown in Figure 6.6.1. Assuming that the order of integration can be reversed, we finally obtain

$$F(s)G(s) = \int_0^\infty e^{-st}\, dt \int_0^t f(t-\tau)g(\tau)\, d\tau, \tag{12}$$

or

$$F(s)G(s) = \int_0^\infty e^{-st} h(t)\, dt$$
$$= \mathcal{L}\{h(t)\}, \tag{13}$$

where $h(t)$ is defined by Eq. (2). This completes the proof of Theorem 6.6.1.