1 Derivation of Lagrange Equations

Consider a particle acted upon by forces X, Y, Z. By Newton's 2nd law:

$$m\ddot{x} = X \tag{1}$$
$$m\ddot{y} = Y \tag{2}$$
$$m\ddot{z} = Z \tag{3}$$

Suppose that:

$$x = x(q_1, q_2, q_3, \ldots, q_n, t) \tag{4}$$
$$y = y(q_1, q_2, q_3, \ldots, q_n, t) \tag{5}$$
$$z = z(q_1, q_2, q_3, \ldots, q_n, t) \tag{6}$$

That is, x, y, z are functions of generalised coordinates $q_1, q_2, q_3, \ldots, q_n$.
And that each coordinate is a function of time: $q_i = q_i(t)$.
It follows that:

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial x}{\partial q_n}\dot{q}_n + \frac{\partial x}{\partial t} \tag{7}$$
$$\dot{x} = \frac{\partial x}{\partial q_i}\dot{q}_i + \frac{\partial x}{\partial t} \tag{8}$$
$$\Rightarrow \dot{x} = f(q_i, \dot{q}_i, t) \tag{9}$$

Where $i = 1, 2, \ldots, n$. Hence, we may write the differential of $\dot{x}$ with respect to the component q_i:

$$\frac{\partial \dot{x}}{\partial \dot{q}_i} = \frac{\partial x}{\partial q_i} \tag{10}$$
$$\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_i} = \dot{x}\frac{\partial x}{\partial q_i} \tag{11}$$
$$\frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial q_i}\right) \tag{12}$$
$$= \ddot{x}\frac{\partial x}{\partial q_i} + \dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_i}\right) \tag{13}$$

Now, if we define:

$$f_i(q_1, q_2, \ldots, q_n, t) = f_i(q_j, t) \equiv \frac{\partial x}{\partial q_i} \tag{14}$$

Then we have:

$$\frac{d}{dt}f_i = \frac{\partial f_i}{\partial q_j}\dot{q}_j + \frac{\partial f_i}{\partial t} \tag{15}$$
$$= \frac{\partial}{\partial q_j}\frac{\partial x}{\partial q_i}\dot{q}_j + \frac{\partial}{\partial t}\frac{\partial x}{\partial q_i} \tag{16}$$
$$= \frac{\partial}{\partial q_i}\left(\frac{\partial x}{\partial q_j}\dot{q}_j + \frac{\partial x}{\partial t}\right) \tag{17}$$

$$= \frac{\partial}{\partial q_i}\dot{x} \tag{18}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial x}{\partial q_i}\right) = \frac{\partial}{\partial q_i}\dot{x} \tag{19}$$

Hence, we have, after substituting (19) into (13):

$$\frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_i}\right) = \ddot{x}\frac{\partial x}{\partial q_i} + \dot{x}\frac{\partial \dot{x}}{\partial q_i} \tag{20}$$

Notice:

$$\frac{d}{d\dot{x}}\left(\frac{1}{2}\dot{x}^2\right)\frac{\partial \dot{x}}{\partial q_i} = \dot{x}\frac{\partial \dot{x}}{\partial q_i} \tag{21}$$

$$\Rightarrow \frac{\partial}{\partial q_i}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\frac{\partial \dot{x}}{\partial q_i} \tag{22}$$

And noting similarly that $\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_i}$; rewrite (20):

$$\ddot{x}\frac{\partial x}{\partial q_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}\dot{x}^2\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}\dot{x}^2\right) \tag{23}$$

Thus, multiplying through by m, and noting that $m\ddot{x} = X$, and doing similar things for Y, Z:

$$X\frac{\partial x}{\partial q_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}m\dot{x}^2\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}m\dot{x}^2\right) \tag{24}$$

$$Y\frac{\partial y}{\partial q_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}m\dot{y}^2\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}m\dot{y}^2\right) \tag{25}$$

$$Z\frac{\partial z}{\partial q_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}m\dot{z}^2\right) - \frac{\partial}{\partial q_i}\left(\frac{1}{2}m\dot{z}^2\right) \tag{26}$$

$$\tag{27}$$

Adding gives:

$$X\frac{\partial x}{\partial q_i} + Y\frac{\partial y}{\partial q_i} + Z\frac{\partial z}{\partial q_i} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_i}\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\right] - \frac{\partial}{\partial q_i}\left[\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\right] \tag{28}$$

But, $\frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = T'$, the kinetic energy of a particle.
Thus:

$$X\frac{\partial x}{\partial q_i} + Y\frac{\partial y}{\partial q_i} + Z\frac{\partial z}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}T'\right) - \frac{\partial}{\partial q_i}\left(T'\right) \tag{29}$$

For a continuous body of particles $p_1, p_2, \ldots$, where $p_i(x_i, y_i, z_i)$. Applying (29) to particle p_j:

$$X_j\frac{\partial x_j}{\partial q_i} + Y_j\frac{\partial y_j}{\partial q_i} + Z_j\frac{\partial z_j}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}T'_j\right) - \frac{\partial}{\partial q_i}\left(T'_j\right) \tag{30}$$

However, $\sum T_i = T =$ total kinetic energy of body.
So, defining Q_i as the generalised component of force:

$$Q_i \equiv \sum_j\left(X_j\frac{\partial x_j}{\partial q_i} + Y_j\frac{\partial y_j}{\partial q_i} + Z_j\frac{\partial z_j}{\partial q_i}\right) \tag{31}$$

Thus:

$$Q_i = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}T - \frac{\partial}{\partial q_i}T \tag{32}$$

Now, if the forces acting on the body are conservative, we have a relation betwen the generalised force Q_i and potential energy function V:

$$Q_i = -\frac{\partial V}{\partial q_i} \tag{33}$$

Where the potential $V(q_i, t)$. Thus, (32) can be rewritten:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} \tag{34}$$

Now, if we define the Lagrangian to be:

$$\mathcal{L} \equiv T - V \tag{35}$$

Thus we can write the lagrangian equation of motion:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{36}$$