Computational methods for uncertainty quantification and identification

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June 11th 2014

3. Stochastic Modeling (MaxEnt) / Solver Solver (Monte Carlo method)

Stochastic Modeling × Stochastic Solver

Stochastic Modeling - Maximum Entropy Principle

Definition of the variables that will be model as random variables (random vectors, random matrices, random processes,..) and construction of the probabilistic model (probability density function).

Stochastic Solver - Monte Carlo Method

Propagation of the uncertainty. Quantification of the response uncertainty. Approximation of the statistics of the response.

How to choose/construct the probabilistic model

Since we need to choose a model, what is the best strategy?

- Just pick up a probabilistic model, for instance, Normal random variable.
- Hypothesis test to choose the probabilistic model.
- Construct the probabilistic model using the MaxEnt Principle.

Information theory

All information can be quantified in a precise manner. Ex. phone signal, text, radio waves, images, all means of communication, etc. (Digital Era)

Shannon's Entropy (1948):

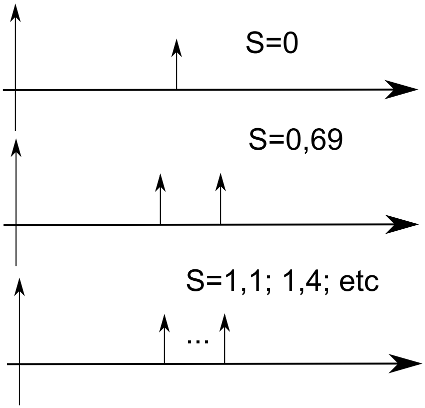
$S = -\sum p_i \ln p_i$,

where $S \in \mathbb{R}^+$ and p_i is the probability of the i-th state.

If $p_i = p$, then $S = \ln n$.

Interpretation: if $P(A) = 1$ (sure event), then $S = 0$ (no uncertainty).

Random field



Entropy

Properties:

- S is a function of $p_1, ..., p_n$
- S is a continuous function of $p_1, ..., p_n$
- S is symmetric with respect to permutations of p_i
- S is a concave function of $p_1, ..., p_n$

Maximum Entropy Principle

"Of all the probability distributions consistent with a given set of information, choose the one with maximum entropy (uncertainty)".

Construction of the probability density function rationally, respecting some properties of interest.

Example: coin toss, which distribution to use? If the only information given is the possibilities of two events (H/T) to occur, the MaxEnt principle yields $P(H) = P(T) = 0.5$

On the board...

General case for discrete distributions. Discrete Uniform example.

General case for continuous distributions. Continuous Uniform example.

Three examples

- If (1) supp=$]-\infty,\infty[$, (2) $E\{X\}=\mu_X$ and (3) $E\{X^2\}=c$, $|c|<\infty$, then MaxEnt yields the NORMAL distribution.
- If (1) supp=$]0,\infty[$, (2) $E\{X\}=\mu_X$, then MaxEnt yields the EXPONENTIAL distribution.
- If (1) supp=$]0,\infty[$, (2) $E\{X\}=\mu_X$ and (3) $E\{1/X^2\}=c$, $|c|<\infty$, then MaxEnt yields the GAMMA distribution.

Spring-force example

Consider the simple example $f = kx$ (took from Soize's course in PUC-Rio 2006).

If the stiffness is random, then $X = f/K$ is random. Since we want the response to have finite variance $E\{1/K^2\} = c$, $|c| < \infty$.

- If $K \sim$ *Normal* it can be verified that $E\{1/K^2\} = c$, $|c| = \infty$. Therefore not a good choice. In addition, there is no physical sense in $k < 0$.
- If $K \sim$ *Exponential* it can be verified that $E\{1/K^2\} = \infty$. Again, not a good choice.
- If $K \sim$ *Gamma* it can be verified that $E\{1/K^2\} = c$, $|c| < \infty$. the best choice.

Two set of springs example

Took from my Qualify (2007).

Last comment on MaxEnt

The MaxEnt Principle can be used to construct probabilistic models of random variables, random vectors, random matrices,...

Although the work might be tough to construct the models, there are many results available in the literature that can be suited for your problem. See Kapur (1992).

Soize (2000) proposed random matrices to model the operators of the system, therefore modeling errors within the model.

Propagation of the uncertainty

Non-intrusive

The deterministic computational model (Finite Element, Finite Difference, or whatever) does not need to be modified for the stochastic analysis.

Intrusive

The deterministic computational model (Finite Element, Finite Difference, or whatever) must be modified/reformulated to include the stochastic analysis.

Propagation of the uncertainty

Spectral Methods

Stochastic Galerkin (intrusive) and stochastic collocation (non-intrusive),...

Monte Carlo Methods

Classical, Latin Hypercube, Quasi-Monte Carlo,...

Integration of a function $h(x)$ in the domain D.

$$I = \int_D h(x)dx$$

If we do $\times \left(\frac{f(x)}{f(x)}\right) = 1$

$$I = \int_D h(x)\frac{f(x)}{f(x)}dx.$$

If $f(x)$ is a probability density function, then $I = E\left\{\frac{h(x)}{f(x)}\right\}$

because $E\{g(X)\} = \int_D g(x)f(x)dx.$

Mean and variance estimators

Mean:

$$E\{X\} \rightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Variance:

$$E\{(X-\mu)^2\} \rightarrow \hat{\sigma}^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(X_i - \hat{\mu})^2.$$

The error of the mean decreases with $1/\sqrt{n}$.

Monte Carlo Integration

If $f(x) = 1/(b-a)$ (Uniform distribution), then $I = E\{h(x)\}(b-a)$.

Which can be approximated by $I \approx \frac{1}{n}\sum_i h(X_i)(b-a)$, where X_i is one observation of $X \sim Unif(a,b)$.

This is a good strategy for high dimension integrals (better than Gauss points, etc.).

$$I \approx \frac{(b_1-a_1)(b_2-a_2)\ldots(b_m-a_m)}{n}\sum^{n} h(X_i).$$

* the choice of the distribution is important for efficiency.

Convergence

Different types:

- In distribution: $\lim_{n\to\infty} F_n(x) = F(x)$
- In probability: $\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$
- Almost sure: $P\left(\lim_{n\to\infty} X_n = X\right) = 1$
- Sure: $\lim_{n\to\infty} X_n(\omega) = X(\omega)$
- Mean: $\lim_{n\to\infty} E|X_n - X|^r = 0$

If $s > r$, then $L^s \quad \rightarrow \quad L^r \quad \rightarrow \quad Prob. \quad \rightarrow \quad Dist.$

$Sure \quad \rightarrow \quad A.Sure \rightarrow \quad Prob. \quad \rightarrow \quad Dist.$

Monte Carlo Loop

Exemplo do deslocamento da viga (MATLAB)... Loop é lento!

* Batou, Ritto and Sampaio (2014).

* Other articles if there is time to.

References

Maximum Entropy

- J. N. Kapur and H. K. Kesavan. Entropy Optimization Principles with Applications. Academic Press, Inc., USA, 1992.

Monte Carlo

- R. Y. Rubinstein. Simulation and the Monte Carlo Method. Series in Probability and Statistics. John Wiley and Sons, New Jersey, USA, 2nd edition, 2007.

Articles

- Batou, A., Ritto, T.G., Sampaio, R., 2014. Entropy propagation analysis in stochastic structural dynamics: application to a beam with uncertain cross sectional area. Computational Mechanics.

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