Computational methods for uncertainty quantification and identification

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About us

- Prof. Daniel Castello, DSc. UFRJ (2004), PosDoc University of Auckland (New Zealand, 2012); Inverse Problems, Prof. Kaipio.

- Prof. Thiago G. Ritto, DSc. Université Paris-Est / PUC-Rio (2010); Stochastic Modeling, Prof. Soize.

* More than 30 papers, related to this field, published in international Journals

Laboratório de Acústica e Vibrações

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Acústica e Vibrações

About us

Research lines:

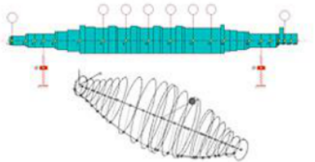
- Stochastic modeling and uncertainty quantification
- Structural dynamics (including influence of the fluid)
- Non-linear dynamics (stability and bifurcations)
- Calibration and model validation
- Damage identification
- Rotordynamics
- Drill-string dynamics

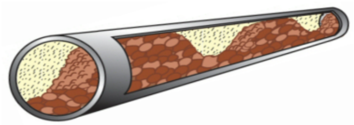
Motivation

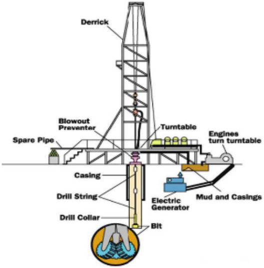
Applications - Decision under uncertainties

Stock market, Justice, everyday decisions, etc.

Mechanical systems.

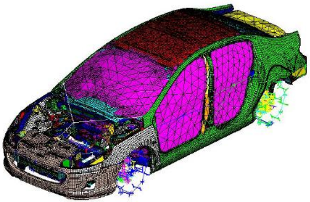


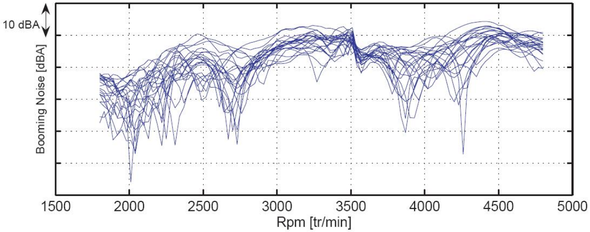




Example: Peugeot car

Measurement of a complex system. Car acoustic response example.





*Durand, J.-F., Soize, C., Gagliardini, L. (2008)

Some recent books

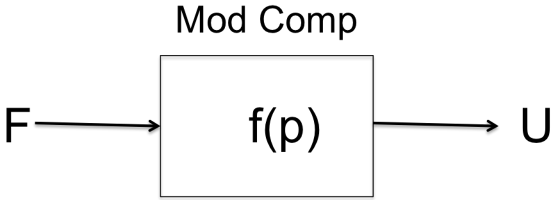
- C. Soize, 2012, Stochastic Models of Uncertainties in Computational Mechanics.
- R.C. Smith, 2012, Uncertainty Quantification: Theory, Implementation, and Applications.
- D. Xiu, 2010, Numerical Methods for Stochastic Computations: A Spectral Method Approach.
- M. Grigoriu, 2012, Stochastic Systems: Uncertainty Quantification and Propagation.
- R.G. Ghanem and P.D. Spanos, 2012 (Dover), Stochastic Finite Elements: A Spectral Approach.
- Kaipio and E. Somersalo, 2005, Statistical and Computational Inverse Problems.

Structure of the course

1. Intruduction to the probability theory, and random number generation with MATLAB
2. Introduction to identification
3. Stochastic Modeling (MaxEnt) / Solver and the Monte Carlo method
4. Minimization techniques, sensitivity analysis, and Bayesian approach

1. Intruduction to the probability theory, and random number generation with MATLAB

Computational Model Uncertainties



- Uncertainties in the forces (input)
- Uncertainties in the parameters ($\mathbf{p}$)
- Uncertainties in the model itself ($f(\cdot)$)
- Consequence –> Uncertainties in the response (ouput)

How to model uncertainties

Probabilistic $\times$ non-probabilistic approaches

- Probability theory (Kolmogorov, 1933)
- Interval theory (Sunuga 1958, Moore 1966)
- Fuzzy sets (Zadeh 1965)
- Possibility theory (Zadeh 1978)
- Info-gap theory (Ben-Haim 1980)
- Etc

How to model uncertainties

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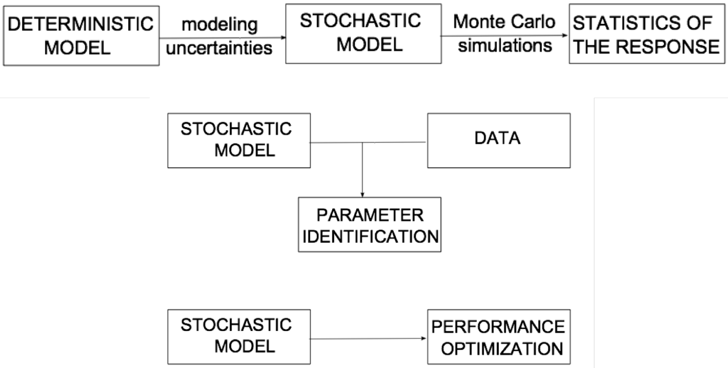
Big Picture



* Ritto, 2010.

Parameter vs. model uncertainties

Parameter uncertainties

Uncertainty of a system parameter, such as stiffness, damping, geometry, etc.

Model uncertainties

Modeling error due to incomplete information and unmodeled phenomena. Ex. simplifications in order to decrease the complexity of the system.

Random vs. epistemic uncertainties

Random uncertainties

Each time we observe the phenomenon, we have a different result (e.g. wind speed, coin toss,...)

Epistemic uncertainties

Lack of knowledgement about a parameter or model (e.g. damping rate, model selection,...)

Numerical error vs. uncertainties

Numerical error

Can (and must) be controlled: finite element approximation, truncation

Uncertainties

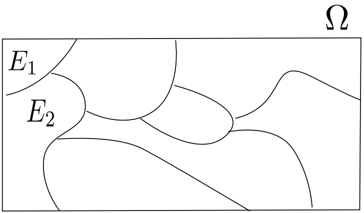
Should be propagate throughout the system

Probability theory

Probability is one way to express the chance of a given (random) event to occur*.

Probability Axioms

1. $P(E) \geq 0$
2. $P(\Omega) = 1$
3. $P(E_1 \bigcup E_2 \bigcup \ldots) = \sum_i P(E_i)$



where E=event, E_i= disjoint events, Ω=random space (set of all possible events)

*intrinsic uncertainty

Probability space is defined by the triple $(\Omega, \mathscr{F}, P)$, where

Ω=random space

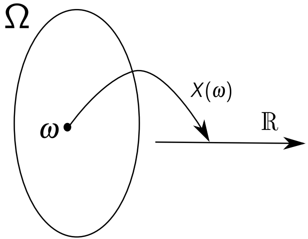
$\mathscr{F}$=σ-algebra (collection of the subsets of the random space)

P=probability measure.

Properties of $\mathscr{F}$:

1. $0 \in \mathscr{F}$
2. If $E \in \mathscr{F}$ then $\overline{E} \in \mathscr{F}$, where $\overline{E} \bigcup E = \Omega$
3. If $E_1, E_2, \ldots \in \mathscr{F}$ then $E_1 \bigcup E_2 \bigcup \ldots \in \mathscr{F}$

Random variable is a function $X : \Omega \mapsto \mathbb{R}$



Example: coin toss, head (H) and tail (T).

We can assign, for instance, $X(H) = 0$ and $X(T) = 1$.

$\Omega\{H, T\}$

$\mathscr{F} = \{\{\}, \{H\}, \{T\}, \{H, T\}\} \quad , \quad (2^\Omega = 2^2 = 4)$

We can assign (must justify!) $P(H) = P(T) = 0,5$

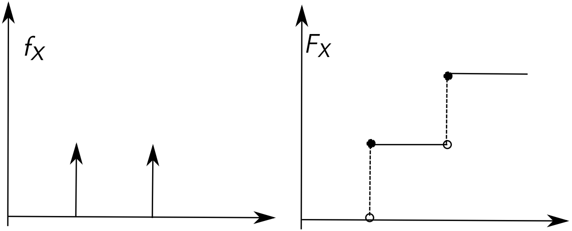
We have $P(\{\}) = 0$ and $P(\{H, T\}) = 1$

Probability distribution and probability density function

Probability distribution: $F_X(x) = P(X < x)$. It is non-decreasing and right continuous.

$\lim_{x \longrightarrow -\infty} F_X(x) = 0$ and $\lim_{x \longrightarrow \infty} F_X(x) = 1$.

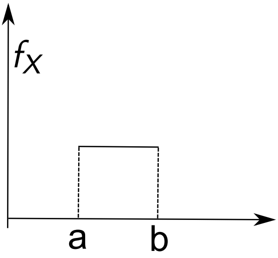
Probability density function: $f_X(x) = \frac{dF_X}{dx}$. Then, $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$.

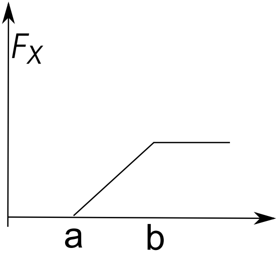


Continuous distribution

Two properties: $f_X(x) \geq 0$ and $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

Uniform probability density function, $Supp = [a, b]$.





Moments of a random variable

First moment: $E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx = cte.$

Second moment: $E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x) dx = cte.$

Third moment: $E\{X^3\} = \int_{-\infty}^{\infty} x^3 f_X(x) dx = cte.$

Mean value: $\mu = E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx = cte.$

Variance: $Var(X) = E\{(X-\mu)^2\} = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) dx = cte.$

Standard deviation: $\sigma(X) = \sqrt{Var(X)} = cte.$

$E\{X\}$ is the mathematical expectation which is a linear operator ($E\{aX\} = aE\{X\}$ and $E\{X+Y\} = E\{X\} + E\{Y\}$).

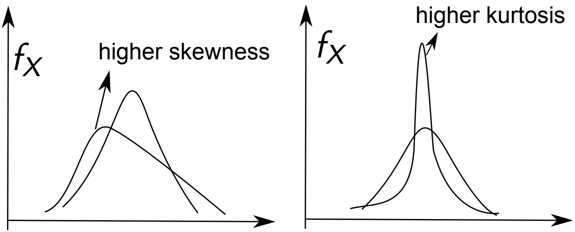
Moments of a random variable

First moment measures the mean.

Second moment measures the dispersion.

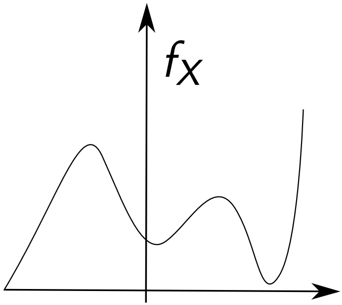
Third moment measures the skewness.

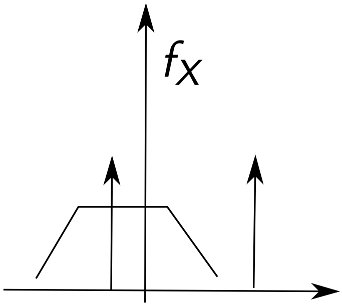
Fourth moment the pickiness (kurtosis).



Probability distribution

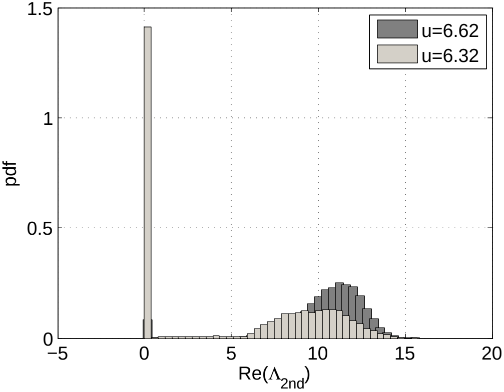
<u>Multi-modal</u> and <u>mixed</u> distributions.





Probability distribution

*Ritto, Soize, Rochinha, Sampaio, 2014, Journal of Fluids and Structures. Probability of flutter stability of a pipe conveying fluid. If $Re(\Lambda_2) > 0$ the system is unstable. We observed a mixed distribution, with dirac delta at $Re(\Lambda_2) = 0$ (stable system).



Probability distribution

Some popular probability distributions:

- Uniform
- Normal
- Gamma
- Rayleigh (wind speed $v = \sqrt{v_x^2 + v_y^2}$)
- Weibull (extreme value)
- Gumbel (extreme value)

Conditional probability and independence

$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$.

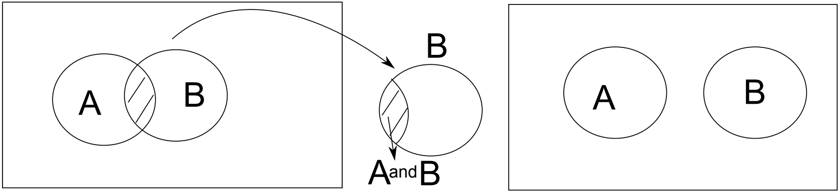
A and B are independent if $P(A|B) = P(A) = \frac{P(A \bigcap B)}{P(B)}$, i.e., knowledgement of B does not affect the probability of A.

Therefore, $P(A \bigcap B) = P(A)P(B)$.

If B_i $\{i = 1, .., n\}$ are disjoint event and $\sum_i B_i = \Omega$

For any event A we have $P(A) = \sum_i P(A \bigcap B_i)$,

which is known as the law of total probability.



P(A)=A/Ret P(B)=B/Ret P(A|B)=(AandB)/B Is A independent of B?

They are <u>not</u> independent, since $P(A) > 0$, $P(B > 0)$ and $P(A \bigcap B) = 0$, hence $P(A \bigcap B) \neq P(A)P(B)$.

Let X and Y be two random variables and $\{X < x\}$ and $\{Y < y\}$, two events.

$P(X < x|Y < y) = \frac{P(X<x \bigcap Y<y)}{P(Y<y)}$.

That is, $F_{X|Y}(x|y) = \frac{F_{XY}(x,y)}{F_Y(y)}$ and $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$ (need proof).

If the two events are independent for any x and y:
$F_{XY}(x,y) = F_X(x)F_Y(y)$ and $f_{XY}(x,y) = f_X(x)f_Y(y)$

Transformation of random variable

$Y = g(X)$. Theorem: $f_Y = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \ldots.$

where $x_1, x_2, \ldots$ are the real roots.

Example: $Y = aX$. In this case we have $Y' = a$ and $X = Y/a$ (one root).

Therefore, $f_Y(y) = \frac{f_X(y/a)}{a}$. If $X \sim Normal(\mu, \sigma)$, then

$$f_Y(y) = \frac{1}{a}\frac{1}{\sqrt{2\pi}\sigma} \exp\left(\left(-\frac{(y/a-\mu)^2}{2\sigma^2}\right)\right).$$

Hence, $Y \sim Normal(a\mu, a\sigma)$.

MATLAB (*TransformacaVA.m* e *TransformacaVA2.m*)

Example: $Y = \exp(X)$. In this case we have $Y' = \exp(X)$ and $X = \ln(Y)$ (one root).

Therefore, $f_Y(y) = \frac{1}{y} f_X(\ln(y))$. If $X \sim Normal(\mu, \sigma)$, then

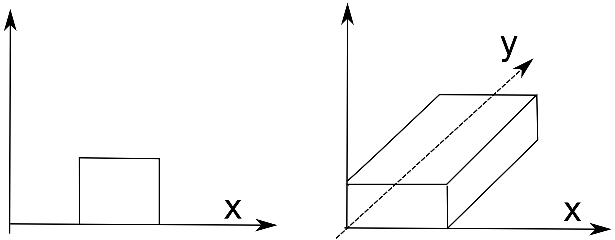
$$f_Y(y) = \frac{1}{y}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln(y)-\mu)^2}{2\sigma^2}\right).$$

Hence, $Y \sim Log-Normal(\mu, \sigma)$ (parameters of the original Normal distribution).

The mean mean and standard deviation are η and τ.
$\mu = \ln\left(\frac{\eta^2}{\sqrt{\tau^2+\eta^2}}\right)$ and $\sigma = \sqrt{\ln\left(\frac{\tau^2}{\eta^2+1}\right)}$.

Note that the support of Y is $]0, \infty[$.

Two random variables



The joint probability density function is f_{XY} and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$.

If X and Y are independent $f_{XY}(x,y) = f_X(x)f_Y(y)$ and $E\{XY\} = E\{X\}E\{Y\}$.

Two random variables

$\{X < x\} \bigcap \{Y < y\} = \{X < x, Y < y\}$ and the joint probability distribution is $F_{XY}(x,y) = P\{X < x, Y < y\}$. Properties:

$F_{XY}(-\infty, y) = 0,\ F_{XY}(x, -\infty) = 0,\ F_{XY}(\infty, \infty) = 1.$

Density function $f_{XY}(x,y) = \partial^2 F_{XY}(x,y) \partial x \partial y$ and

$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\alpha, \beta) d\alpha\beta$

$P((X,Y) \int D) = \int_D f_{XY}(x,y) dxdy.$

Marginal distributions:

$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy$ and $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y) dx$

Two random variables

Joint probability density function of two Normal random variables:

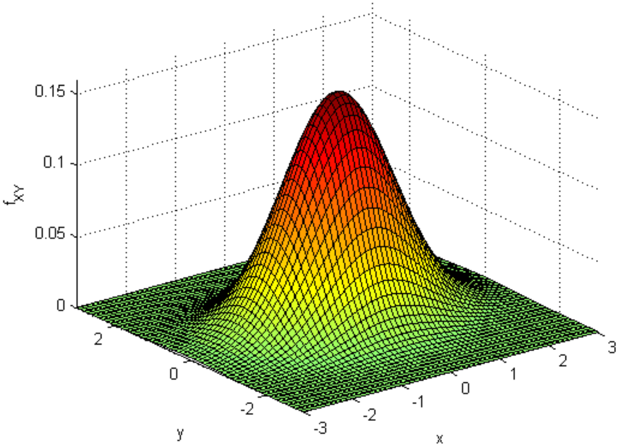
$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right)$$

where μ_x, μ_y, σ_x and σ_y are the mean and standard deviation of the random variables. The coefficient ρ measures the correlations between the two random variables and $|\rho| < 1$.

If f_{XY} is a joint Normal probability density function, then the marginal probability density functions f_X and f_Y are Normal (the other way around is not true.

Two random variables

$\rho = 0.5$



Covariance

$C_{XY} = E\{(X-\mu_X)(Y-\mu_Y)\}$, note that $C_{XX} = Var\{X\} = E\{(X-\mu_X)^2\}$.

$C_{XY} = E\{XY\} - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y$.

$C_{XY} = E\{XY\} - \mu_X\mu_Y$.

If the covariance is zero $E\{XY\} = \mu_X\mu_Y$.

If the random variables are independent $C_{XY} = \mu_X\mu_Y - \mu_X\mu_Y = 0$.

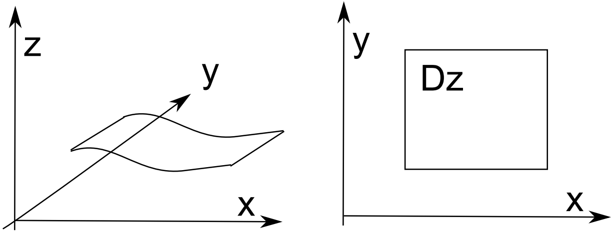
If X and Y are independent they are uncorrelated, the other way around is not true.

Function of two random variables

$Z = g(X, Y)$.

$F_Z(z) = P(Z \leq z) = P(g(X < Y) \leq z)$

$= P((X, Y) \in D_Z) = \int_{D_Z} f_{XY}(x, y) dxdy$.



Function of two random variables

Examples:

$Z = X + Y \sim$ Triangular, if X and Y are independent Uniform.

$Z = \frac{X}{Y} \sim$ Cauchy, if X and Y are independent Normal.

$Z = X^2 + Y^2 \sim$ Exponential, if X and Y are independent Normal.

$Z = \sqrt{X^2 + Y^2} \sim$ Rayleigh, if X and Y are independent Normal.

$$(X,Y) \mapsto \begin{cases} Z = g(X,Y) \\ W = h(X,Y) \end{cases}$$

Applying the theorem

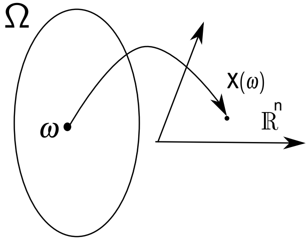
$$f_{ZW}(z,w) = \sum_i \frac{1}{|J(x_i,y_i)|} f_{XY}(x_i,y_i).$$

where (x_i, y_i) is the i-th solution of the system and the Jacobian is given by:

$$J(x_i,y_i) = \left| \begin{array}{cc} \partial g/\partial x & \partial g/\partial y \\ \partial h/\partial x & \partial h/\partial y \end{array} \right|_{(x_i,y_i)}.$$

Sequence of random variables (Random Vector)

$\mathbf{X} : \Omega \mapsto \mathbb{R}^n$



$\mathbf{X} = (X_1, X_2, ..., X_n)^T$ and $P(\mathbf{X} \in D) = \int_D f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$

$f_{\mathbf{X}}(\mathbf{x}) = f_{\mathbf{X}}(x_1, x_2, ..., x_n) = \frac{\partial F_{\mathbf{X}}(x_1, x_2, ..., x_n)}{\partial x_1 \partial x_2 ... \partial x_n}$

$F_{\mathbf{X}}(\mathbf{x}) = P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n\}$

Marginal probability density functions:

$f_{X_1}(x_1) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, x_2, ..., x_n) d_{x_2} d_{x_3} ... d_{x_n}$

If $n = 4$, $f_{X_1 X_3}(x_1, x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\mathbf{X}}(x_1, x_2, x_3, x_4) d_{x_2} d_{x_4}$

If random variables are independent
$F_{\mathbf{X}}(x_1, x_2, ..., x_n) = F_{X_1}(x_1) F_{X_2}(x_2) ... F_{X_n}(x_n)$ and

$f_{\mathbf{X}}(x_1, x_2, ..., x_n) = f_{X_1}(x_1) f_{X_2}(x_2) ... f_{X_n}(x_n)$

Multivariate Normal probability density function

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1}(\mathbf{x}-\mu)\right)$$

where Σ is the covariance matrix of $\mathbf{X}$ and μ is the mean.

$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)$ for $\mu = \mathbf{0}$ and $\Sigma = \mathbf{I}$.

$\Sigma = E\{(\mathbf{X}-\mu)(\mathbf{X}-\mu)^T\} = E\{\mathbf{X}\mathbf{X}^T\} - \mu\mu^T$ or

$\Sigma_{ij} = E\{(x_i - \mu_i)(x_j - \mu_j)\}$, hence

$$\Sigma = \begin{bmatrix} E\{(x_1-\mu_1)(x_1-\mu_1)\} & \ldots & E\{(x_1-\mu_1)(x_n-\mu_n)\} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & E\{(x_n-\mu_n)(x_n-\mu_n)\} \end{bmatrix}$$

where $E\{(x_1-\mu_1)(x_1-\mu_1)\} = \sigma_1^2$ and $E\{(x_1-\mu_1)(x_2-\mu_2)\} = \rho_{12}\sigma_1\sigma_2$.

Covariance and (auto)correlation

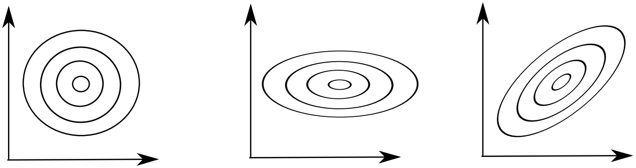
For two random vectors $\mathbf{X}$ and $\mathbf{Y}$

$Cov(\mathbf{X},\mathbf{Y}) = E\{(\mathbf{X}-\mu_{\mathbf{X}})(\mathbf{Y}-\mu_{\mathbf{Y}})^T\}$

(Auto)correlation matrix:

$R = E\{\mathbf{X}\mathbf{X}^T\}$ or $R_{ij} = E\{X_i X_j\}$

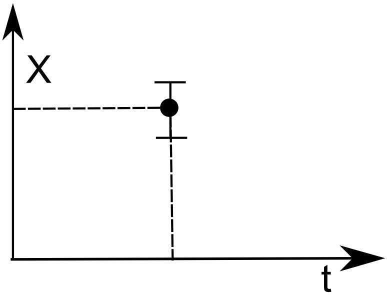
Note that $\sum_{ij} = R_{ij} - \mu_i \mu_j$.



Example: uncorrelated with $\sigma_1 = \sigma_2$; uncorrelated with $\sigma_1 > \sigma_2$; positive correlated with $\sigma_1 > \sigma_2$.

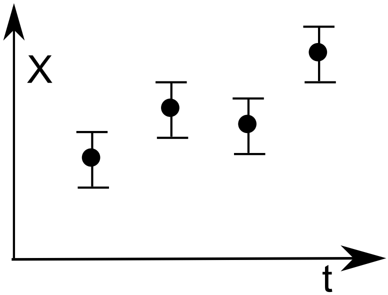
X is a scalar random variable $X : \Omega \mapsto \mathbb{R}$.

$(\Omega, \mathscr{F}, P)$, with f_X, $E\{X\}$, σ_X^2, etc.



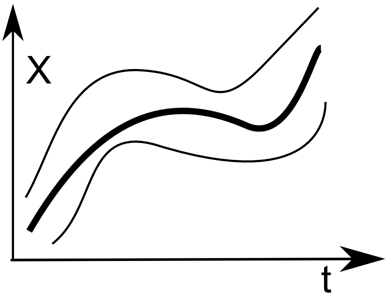
$\mathbf{X}_n$ is a random vector $\mathbf{X}_n : \Omega \mapsto \mathbb{R}^n$.

$(\Omega, \mathscr{F}, P)$, with f_X, $E\{X\}$, σ_X^2, $C_{ij} = E\{(X_i - \mu_i)(X_j - \mu_j)\}$, etc.



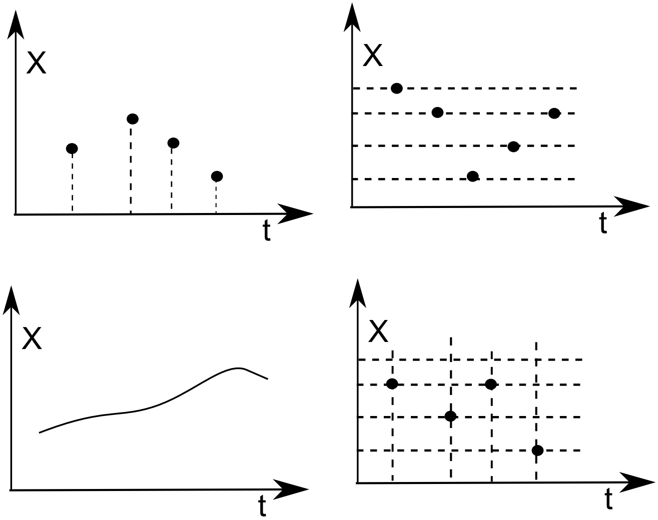
$X(t)$ is a stochastic process $\{X_t; t \in T\}$; collection of random variables indexed by t.

$(\Omega, \mathscr{F}, P)$, with $F_{X_t}(x;t) = P(X(t) \leq x)$, f_{X_t}, $E\{X_t\}$, $R_{t_i t_j}$, etc.



Types of stochastic Processes

Continuous/discrete with continuous/discrete time



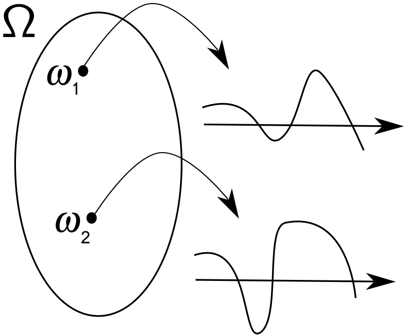
VIDEOS (wind turbine, turbulence)

Stochastic Process

Mean: $\mu(t) = E\{X(t)\}$

Autocorrelation: $R(t_1, t_2) = E\{X(t_1)X(t_2)\}$

Mean power: $R(t,t) = E\{X(t)^2\}$



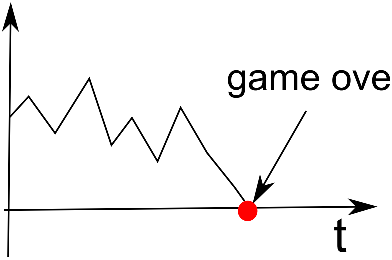
Examples:

Deterministic signal: $X(t) = f(t)$.

Voltage of an AC generator: $X(t) = R\cos(\omega t + \phi)$, with R and ϕ random variables.

Random walk $X_i = \sum_{i=1}^{n} Z_i$, with Z_i a discrete random variable $\{-1, 1\}$.

Gambler's ruin: a theorem guarantees that the process will pass through zero (if $P(-1) \geq 50\%$).



Black-Scholes(1973): stochastic modeling of the stock market using the idea of random walk.

The Casinos will always have a big probability of winning in the long run because $P(1) \geq 80\%$.

MATLAB (*RW.m*)

Stationary process (implies steady state)

Strict sense: joint probability density function f_{X_t} is constant with respect to any displacement in time (for any n and Δt). Very strong condition! Hard to verify!

$f_{X_1 X_2 \dots X_n}(x_1, x_2, ..., x_n; t_1, t_2, ..., t_n) =$
$f_{X_1 X_2 \dots X_n}(x_1, x_2, ..., x_n; t_1 + \Delta t, t_2 + \Delta t, ..., t_n + \Delta t)$.

Weak or wide sense: $n = 2$, for any Δt.

$f_{X_1 X_2}(x_1, x_2; t_1, t_2) = f_{X_1 X_2}(x_1, x_2; t_1 + \Delta t, t_2 + \Delta t)$.

Note that $t1 + \Delta t - t_2 = 0$ and $t1 + \Delta t - t_1 = \Delta t = \tau$. Hence

$f_{X_1 X_2}(x_1, x_2; t_1, t_2) = f_{X_1 X_2}(x_1, x_2; \tau)$ and

$\mu(t) = cte$ and $R(t_1, t_2) = R(\tau)$, with $E\{X(t)^2\} = R(t, t) = R(0) = cte$.

Autocorrelation function

Examples:

$X(t) = A\cos(\omega t)$, then, $R(t_1, t_2) = E\{A^2\}\cos(\omega t_1)\cos(\omega t_2)$.

Exponential function $R(t_1, t_2) = R(\tau) = e^{-|\tau|/b}$.

Triangular function $R(t_1, t_2) = R(\tau) = 1 - d|\tau|$.

$R(\tau) = \frac{\sigma^2 \sin(\omega_c \tau)}{\omega_c \tau} = \sigma^2 \text{sinc}(\omega_c \tau)$.

MATLAB (*autocorrelacao.m* e *Autocorrelacao2.m*).

Power spectral density

PSD: $S_X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_X(\tau)e^{-i\omega\tau}d\tau$

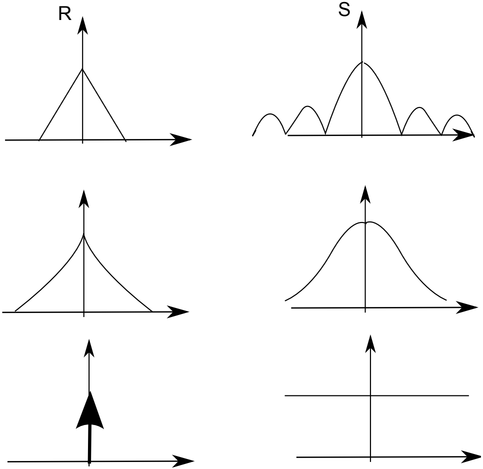
Consequently: $R_X(\tau) = \int_{-\infty}^{\infty} S_X(\omega)e^{i\omega\tau}d\omega$.

Estimator:

$\widehat{S}_X(\omega, T, n) = \frac{1}{2\pi Tn}\sum_{k=1}^{n} |\mathscr{F}(X_k(t, T))|^2$.

i.e., mean of the square of the Fourier transform of the process $X(t)$. In which $\mathscr{F}(\cdot)$ is the Fourier transform, T is the time interval, n is the sample number.

Power spectral density



$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$.

In the frequency domain: $\widehat{x}(\omega) = h(\omega)\widehat{f}(\omega)$,

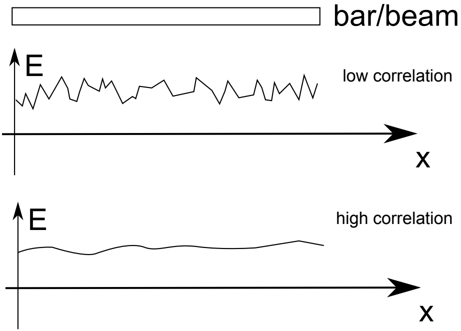
with frequency response function $h(\omega) = \frac{1}{-\omega^2 + ic\omega + k}$.

If $F(t)$ is a stationary random process with $S_F(\omega)$ known:

$S_X(\omega) = |h(\omega)|^2 S_F(\omega)$.

Random field

$\{X_s, s \in S\}$, it is s-indexed instead of t-indexed.



How to generate random variables

It is very hard (if not impossible) to generate random variables!

A popular pseudo-random number generator is the 'Linear Congruential Generator'.

$X_i = mod(aX_{i-1} + c, m)$,

where X_o is the seed, a is the multiplicative factor, c is the increment, m is the modulus and mod is the rest of the division.

For $X_o = 1$, $a = 13$, $m = 31$ and $c = 0$.

$X_1 = 13$, $X_2 = 24$, $X_3 = 27$,...

And if we devide by m, $Y = X/m \sim UNIF(0, 1)$.

IBM (1966) $a = 65539$, $m = 2^{31}$ and $c = 0$.

MATLAB (1988) $a = 7^5$, $m = 2^{31} - 1$ and $c = 0$. The sequence repeats every $(m-1)$ samples.

MATLAB (1997): TWISTER with period $2^{19937} - 1$

Check: periodicity, correlation and convergence.

From the generation of $rand \sim UNIF(0,1)$:

$$U = rand * (b - a) + a \sim UNIF(a,b)$$

$$Z_1 = \sqrt{-2\log(U_1)}\sin(2\pi U_2) \sim Normal(0,1)$$

$$Z_2 = \sqrt{-2\log(U_1)}\cos(2\pi U_2) \sim Normal(0,1)$$

$$X = m + \sigma Z \sim Normal(m,\sigma)$$

To generate random variables from other pdf's: inverse transform method, rejection method, etc.

MATLAB (*ExemploUNIFORME.m* e *ExemploNORMAL.m*)

How to generate correlated random variables

If $X \sim Normal(\mu_X, \sigma_X)$, then $Y = aX + b \sim Normal(a\mu_X, a^2\sigma_X^2)$.

If $\mathbf{X} \sim Normal(\mu_{\mathbf{X}}, \Sigma_{\mathbf{X}})$, then
$\mathbf{Y} = [A]\mathbf{X} + \mathbf{b} \sim Normal([A]\mu_X, [A]\Sigma[A]^T)$.

Let $Z \sim Normal(0, 1)$ and $\mathbf{Z} \sim Normal(\mathbf{0}, \mathbf{I})$, then we can generate X and $\mathbf{X}$

$X = \mu_X + \sigma Z$ and $\mathbf{X} = \mu_{\mathbf{X}} + [\sigma]\mathbf{Z}$

where $\Sigma = [\sigma][\sigma]^T$ or $\Sigma = [\sigma][\sigma]$. We still need to compute the deterministic matrix $[\sigma]$.

How to generate correlated random variables

Spectral theorem (symmetric matrix):

$\Sigma = [Q][\Lambda][Q]^T = [\sigma][\sigma]^T$, therefore $[\sigma] = [Q][\Lambda]^{1/2}$.

Cholesky decomposition (positive definite matrix).

$\Sigma = [\sigma'][\sigma']^T$, therefore $[\sigma']$ is a lower triangular matrix.

Singular value decomposition (symmetric matrix):

$\Sigma = [V][\Lambda]^2[V]^T = [\sigma''][\sigma'']$, therefore $[\sigma''] = [V][\Lambda][V]^T$.

MATLAB (*correlacao.m*)

References

Probability theory

- A. Papoulis. Probability, Randon Variables, and Stochastic Processes. McGraw Hill, 2002.

Random vibrations

- S. H. Crandall and W. D. Mark. Random Vibrations in Mechanical Systems. Academic Press, Inc., 1963.
- P. H. Wirsching, T. L. Paez, and K. Ortiz. Random Vibrations: Theory and Practice. Dover Publications, Inc., New York, USA, 2006.

Articles

- Ritto, T.G., Soize, C., Rochinha, F.A., Sampaio, R., 2014. Dynamic stability of a pipe conveying fluid with an uncertain computational model. Journal of Fluids and Structures
- Ritto, T.G., Soize, C., Sampaio, R., 2009. Non-linear dynamics of a drill-string with uncertain model of the bit-rock interaction International Journal of Non-Linear Mechanics

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