Chapter 6

Plane Problems

THERE ARE SOME SITUATIONS IN WHICH THE SPECIFIC CHARACTERISTICS OF THE PROBLEM ALLOW THE A REDUCTION IN ITS COMPLEXITY, TRANSFORMING THE ORIGINAL PROBLEM IN ONE WHICH IS PRONE TO BE SOLVED THROUGH ANALYTICAL MEANS. THAT TYPICALLY OCCURS DUE TO AXISYMMETRY OR "TWO-DIMENSIONALITY" (PLANE PROBLEMS), WHICH WILL BE ADDRESSED IN THIS CHAPTER

6.1) PLANE STRAIN

CONSIDER A LONG ("INFINITELY") PRISMATIC BODY LOADED BY BODY FORCES AND TRACTIONS ON THE LATERAL SURFACES NOT DEPENDING ON THE Z COORDINATE (WHICH IS PARALLEL TO THE "LONG" SIDE OF THE BODY) AND HAVING NO COMPONENT IN THAT DIRECTION. THEN A REASONABLE HYPOTHESIS TO BE ASSUMED ABOUT THE DISPLACEMENT FIELD WOULD BE.

$$u_x = f(x, y) \; ; \quad u_y = g(x, y) \quad \text{AND} \quad u_z = 0$$

THE DEFORMATION ASSOCIATED TO THIS KIND OF APPROXIMATION IS REFFERED TO AS PLANE STRAIN. MOREOVER, THE ELASTICITY PROBLEM IS REDUCED TO A 2-D FORMULATION DEFINED IN THE CROSS SECTION (NORMAL TO THE DIRECTION Z) REGION.

THE STRAIN FIELD CORRESPONDS TO

$$\varepsilon_x = \frac{\partial u_x}{\partial x}; \quad \varepsilon_y = \frac{\partial u_y}{\partial y}; \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

AND

$$\varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

WHICH IMPLIES BY APPLYING HOOKE'S LAW

$$\sigma_x = \lambda\,(\varepsilon_x + \varepsilon_y) + 2\mu\,\varepsilon_x$$

$$\sigma_y = \lambda\,(\varepsilon_x + \varepsilon_y) + 2\mu\,\varepsilon_y$$

$$\sigma_{xy} = 2\mu\,\varepsilon_{xy}$$

$$\sigma_z = \lambda\,(\varepsilon_x + \varepsilon_y) \quad \left(\text{IMPORTANT} \neq 0 \text{ AND } \sigma_z = \nu\,(\sigma_x + \sigma_y)\right)$$

$$\sigma_{xz} = \sigma_{yz} = 0$$

THUS, THE EQUILIBRIUM EQUATION REDUCES TO

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + b_y = 0$$

AND THEN

$$\mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) + (\lambda + \mu) \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + b_x = 0$$

$$\mu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) + (\lambda + \mu) \frac{\partial}{\partial y} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + b_y = 0$$

AND THE COMPATIBILITY EQUATIONS ARE NOW:

$$\frac{\partial^2 \varepsilon_x}{\partial x^2} + \frac{\partial^2 \varepsilon_y}{\partial y^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

WHICH IMPLIES

$$\nabla^2 (\sigma_x + \sigma_y) = - \frac{1}{1 - \nu} \left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} \right)$$

WHERE ∇^2 STANDS FOR $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

6.2) PLANE STRESS

NOW ADMIT THAT THE BODY IS BOUNDED (IN Z DIRECTION) BY TWO PARALLEL PLANES SEPARATED BY A SMALL DISTANCE (IF COMPARED TO THE OTHER TWO DIMENSIONS). SO, THE MODEL IS SUCH THAT IT CAN BE GENERATED BY A PLANAR REGION (PARALLEL TO THE XY PLANE) AND LIMITED (IN Z DIRECTION) BY THE PLANES $Z = \pm h$. AND ALSO ASSUME THAT AT $Z = \pm h$ $\sigma_z = \tau_{xz} = \tau_{yz} = 0$. AS, IN THE Z DIRECTION, THE BODY IS SUPPOSED TO BE THIN, THESE STRESS COMPONENTS ARE EXPECTED TO EXPERIMENT LITTLE VARIATION AND THEREFORE THEY ARE ASSUMED TO VANISH ACROSS THE BODY. MOREOVER, FOLLOWING SIMILAR ARGUMENTS, THE OTHER STRESS COMPONENTS ARE SUPPOSED NOT TO VARY ALONG THE Z DIRECTION. THOSE RELATIONS ARE SUMMARIZED AS

$$\sigma_x = f(x,y); \quad \sigma_y = g(x,y); \quad \tau_{xy} = h(x,y)$$

AND $$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

THIS IS THE SO CALLED PLANE STRESS STATE OF AN ELASTIC BODY. IN ORDER TO MAINTAIN THE STRESS FIELD INDEPENDENT OF z AS A FEASIBLE HYPOTHESIS THE EXTERNAL FORCES IN THE z DIRECTION SHOULD VANISH (WHY?). MOREOVER THE OTHER COMPONENTS OF THE EXTERNAL FORCES ARE SUPPOSED NOT TO VARY ALONG THE z DIRECTION OR DISTRIBUTED IN A SYMMETRIC WAY ABOUT THE MIDPLANE THROUGH THE THICKNESS, SUCH THAT THEY CAN BE REPRESENTED BY AVERAGES (REGARDING THE z DIRECTION). IN THAT SENSE, THE PLANE STRESS PROBLEM CAN BE USED AS A MODEL TO THE IN-PLANE DEFORMATION OF THIN ELASTIC PLATES.

NOW, BY USING HOOKE'S LAW

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$$

$$\longrightarrow \varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\,\tau_{xy} \; ; \quad \varepsilon_{xz} = \varepsilon_{yz} = 0$$

THEREFORE:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial u_y}{\partial y} \quad ; \quad \varepsilon_z = \frac{\partial u_z}{\partial z}$$

AND

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = 0$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0$$

WHICH SAYS:

u_y AND u_x ARE FUNCTIONS OF z!

(WE DON'T HAVE A PLANAR PROBLEM)

AND MORE u_z CAN VARY LINEARLY IN z.

↳ LET US FORGET THIS FOR THE MOMENT AND ANALYZE ITS CONSEQUENCES LATER

THE EQUILIBRIUM EQUATIONS REDUCE TO THE IDENTICAL FORM AS THE ONE OF PLANE STRAIN

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0$$

WHERE b_x AND b_y ARE FUNCTIONS OF X AND Y. THEN

$$\mu \nabla^2 u_x + \frac{E}{2(1-\nu)} \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + b_x = 0$$

$$\mu \nabla^2 u_y + \frac{E}{2(1-\nu)} \frac{\partial}{\partial y} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) + b_y = 0$$

REMARK: NOTE THAT THESE EQUATIONS ARE SIMILAR TO THOSE OF PLANE STRAIN, BUT NOT EQUAL.

REMEMBER ALSO THAT u_x AND u_y MAY VARY WITH z, SO THIS EQUATION REQUIRES 3-D BOUNDARY CONDITIONS

NOW, HEADING AGAIN TO THE COMPATIBILITY EQUATIONS:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

$$\Downarrow$$

$$\nabla^2 (\sigma_x + \sigma_y) = -(1+\nu) \left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} \right)$$

REMEMBER, THE SOLUTION OF THE PROBLEM MUST STILL SATISFY THE OTHER TWO COMPATIBILITY EQUATIONS THAT, DIRECTLY, INVOLVE THE NORMAL DEFORMATION ε_z AND THE VARIATION OF THE PLANAR DEFORMATION $\{\varepsilon_x, \varepsilon_y, \varepsilon_{xy}\}$ WITH RESPECT TO z. USUALLY, THOSE EQUATIONS ARE NOT CONSIDERED, RENDERING THE PLANE STRESS PROBLEM AS ONLY AN APPROXIMATION OF THE INITIAL MODELING.

ONE WAY OF POTENTIALLY IMPROVING THE ACCURACY OF THE APPROXIMATED SOLUTION OBTAINED BY USING THE PLANE STRESS MODELING RELIES ON THE USE OF AVERAGES ALONG THE z-DIRECTION, e.g.:

$$\bar{\phi}(x,y) = \frac{1}{2h} \int_{-h}^{h} \phi(x,y,z)\, dz$$

TAKING INTO CONSIDERATION ALL THE PREVIOUS HYPOTHESIS ABOUT LOADINGS IN Z-DIRECTION

$$u_z(x,y,z) = -u_z(x,y,-z)$$

$$\left(\rightarrow \quad u_z(x,y,0) = 0\right)$$

THUS: $$\bar{u}_z = \frac{1}{2h}\int_{-h}^{h} u_z(x,y,z)\, dz = 0$$

AND

$$\sigma_z(x,y,\pm h) = \sigma_{xz}(x,y,\pm h) = \sigma_{yz}(x,y,\pm h) = 0$$

WHICH, IN TURN, IMPLIES

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

$$\Downarrow$$

$$\frac{\partial \sigma_z}{\partial z}(x,y,\pm h) = 0$$

$\hookrightarrow$ $\sigma_z \approx 0$ ACROSS THE BODY

So, THE PLANE STRESS PROBLEM IS REFORMULATED, GIVEN RISE TO THE FOLLOWING SET OF EQUATIONS:

$$\bar{u}_x = f(x,y)\ ;\ \bar{u}_y = g(x,y) \text{ AND } \bar{u}_z = 0$$

$$\bar{\sigma}_z = \bar{\sigma}_{xz} = \bar{\sigma}_{yz} = 0$$

$$\bar{\sigma}_x = \lambda^* (\bar{\varepsilon}_x + \bar{\varepsilon}_y) + 2\mu \bar{\varepsilon}_x$$

$$\bar{\sigma}_y = \lambda^* (\bar{\varepsilon}_x + \bar{\varepsilon}_y) + 2\mu \bar{\varepsilon}_y$$

$$\bar{\sigma}_{xy} = 2\mu \bar{\varepsilon}_{xy}$$

$$\bar{\varepsilon}_z = -\frac{\lambda}{\lambda + 2\mu} (\bar{\varepsilon}_x + \bar{\varepsilon}_y)$$

WHERE $\lambda^* = \dfrac{2\lambda\mu}{\lambda + 2\mu}$. THE ABOVE EQUATIONS WITH THE AID OF THE EQUILIBRIUM EQUATION AND THE COMPATIBILITY RELATIONS LEAD TO THE FOLLOWING MATHEMATICAL PROBLEMS:

DISPLACEMENTS

$$\mu \nabla^2 \bar{u}_x + (\lambda^* + \mu) \frac{\partial}{\partial x} \left(\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} \right) + \bar{b}_x = 0$$

$$\mu \nabla^2 \bar{u}_y + (\lambda^* + \mu) \frac{\partial}{\partial y} \left(\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} \right) + \bar{b}_y = 0$$

OR

STRESS

$$\nabla^2(\bar{\sigma}_x + \bar{\sigma}_y) = -\frac{2(\lambda^* + \mu)}{\lambda^* + 2\mu}\left(\frac{\partial \bar{b}_x}{\partial x} + \frac{\partial \bar{b}_y}{\partial y}\right)$$

REMARK: MANY APPLICATIONS INVOLVE SPECIFIC GEOMETRIES THAT ARE EASIER HANDLED BY ADOPTING NON CARTESIAN COORDINATES. LET US CHECK ON THE FINAL EXPRESSIONS WHEN POLAR COORDINATES ARE APPLIED

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

HOOKE'S LAW:

PLANE STRAIN

$$\sigma_r = \lambda(\varepsilon_r + \varepsilon_\theta) + 2\mu\,\varepsilon_r$$

$$\sigma_\theta = \lambda(\varepsilon_r + \varepsilon_\theta) + 2\mu\,\varepsilon_\theta$$

$$\sigma_z = \lambda(\varepsilon_r + \varepsilon_\theta) = \nu(\sigma_r + \sigma_\theta)$$

$$\sigma_{r\theta} = 2\mu\,\varepsilon_{r\theta}\,, \quad \sigma_{\theta z} = \sigma_{rz} = 0$$

PLANE STRESS

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = \frac{-\nu}{(1+\nu)}(\varepsilon_r + \varepsilon_\theta)$$

$$\varepsilon_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta}\,, \quad \varepsilon_{\theta z} = \varepsilon_{rz} = 0$$

EQUILIBRIUM EQUATIONS

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{(\sigma_r - \sigma_\theta)}{r} + b_r = 0$$

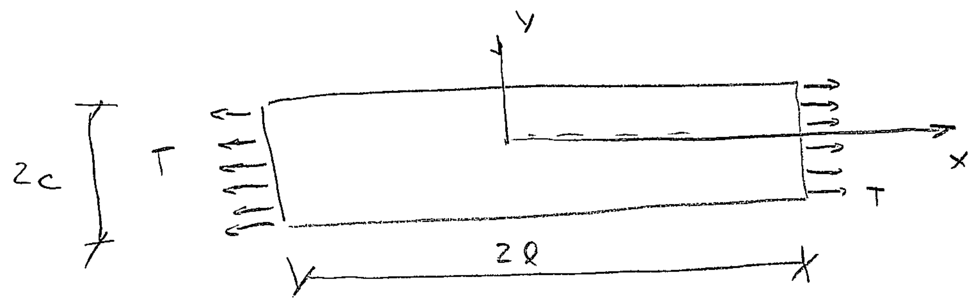
$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + 2\frac{\sigma_{r\theta}}{r} + b_\theta = 0$$

THE FINAL FORM OF THE MATHEMATICAL PROBLEMS CAN BE EASLY ACHIEVED BY COMBINING THE RELATIONS ABOVE

6.3 - EXAMPLES

(1) UNIAXIAL TENSION

CONSIDER A 2D PLANE STRESS (THIN SHEET) CASE OF A LONG RECTANGULAR BEAM UNDER UNIFORM TENSION AS DEPICTED BELLOW



BOUNDARY CONDITIONS:

$$\sigma_x (x = \pm l ; y) = T$$

$$\sigma_y (x ; y \pm c) = 0$$

$$\tau_{xy} (x = \pm l ; y) = 0 \quad \text{AND} \quad \tau_{xy} (x ; y \pm c) = 0$$

P.S.: YOU COULD THINK ON σ_{zi} COMPONENTS, BUT THEY HAVE ALREADY BEEN CONSIDERED WHEN THE PLANE STRESS SOLUTION WAS ASSUMED.

SO, IMMEDIATELY ONE CAN GUESS $\sigma_x = T$; $\sigma_y = \tau_{xy} = 0$ AS A SOLUTION FOR THE ENTIRE BEAM. IT IS EASY TO CHECK THAT IT SATISFIES THE EQUILIBRIUM EQUATION. LET US GO IN A DIFFERENT WAY, JUST TO GET MORE FAMILIAR WITH THE USE O AIRY STRESS FUNCTION. REMEMBER THAT THE STRESS FORMULATION CAN BE HANDLED BY THE INTRODUCTION OF A POTENTIAL ϕ WHICH WAS DEFINED IN pg. 5.12 AND SATISFIES THE BI-HARMONIC EQUATION. CONSIDERING THAT AT THE BOUNDARIES σ_x IS CONSTANT ALONG Y-DIRECTION, A GOOD GUESS FOR ϕ SEEMS TO BE

$$\underbrace{\phi = A}_{\text{CONSTANT}} y^2$$

(NOTE THAT THE SECOND DERIVATIVE OF ϕ WITH RESPECT TO Y IS CONSTANT)

WHICH IS CONFIRMED BY

$$\sigma_x = 2A \;, \; \sigma_y = 0 \;; \; \tau_{xy} = 0$$

AND THEN, APPLYING THE BOUNDARY CONDITION

$$A = \frac{T}{2}$$

AND NOTE THAT OUR FIRST GUESS TO THE STRESS FIELD IS RETRIEVED.

NOW, DEPARTING FROM THE OBTAINED STRESS FIELD, WE WISH TO DETERMINE THE DISPLACEMENT FIELD.

HOOKE'S LAW
$$\begin{cases} \dfrac{\partial u_x}{\partial x} = \varepsilon_x = \dfrac{1}{E}\left(\sigma_x - \nu\sigma_y\right) = \dfrac{T}{E} \\[2ex] \dfrac{\partial u_y}{\partial y} = \varepsilon_y = \dfrac{1}{E}\left(\sigma_y - \nu\sigma_x\right) = -\nu\dfrac{T}{E} \end{cases}$$

THEN:

$$u_x = \frac{T}{E}x + f(y)$$

(NOTE THAT WE HAVE IGNORED A POTENTIAL DEPENDENCY ON z -DIRECTION)

$$u_y = -\nu\frac{T}{E}y + g(x)$$

BUT WE STILL HAVE $\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0$

$$\longrightarrow \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} = 0$$

$$\hookrightarrow f(y) = -\omega y + u_0$$

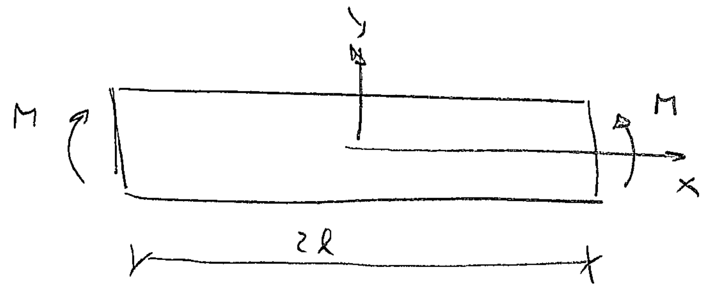
$$g(x) = \omega x + v_0$$

CONSTANT VALUES

$\underbrace{}$

RIGID BODY MOTION

(2) PURE BENDING OF A BEAM



ONCE AGAIN RELYING ON THE SAINT-VENANT PRINCIPLE :

BOUNDARY CONDITIONS

$$\sigma_y(x, y = \pm c) = 0 \, , \quad \tau_{xy}(x, y = \pm c) = 0 \, ; \quad \tau_{xy}(x = \pm l, y) = 0$$

AND

$$\underbrace{\int_{-c}^{c} \sigma_x(x \pm l, y)\, dy = 0}_{\Downarrow} \quad ; \quad \int_{-c}^{c} \sigma_x(x = \pm l, y)\, y \, dy = -M$$

NORMAL APPLIED

FORCE

HOW TO GUESS ABOUT ϕ? NOTE THAT IF σ_x AT $x = \pm l$

IS CONSTANT THEN M MUST VANISH, THUS LET CHECK ON

$\phi = A\, y^3$ (LEADING TO σ_x LINEAR AT THE

BOUNDARIES)

P.S: THE ABOVE INTEGRALS ARE INDEED $\int_{-t}^{t}\int_{-c}^{c} \sigma_x \, dy\, dz$, BUT

AS σ_x DOES NOT DEPEND ON z WE CAN TAKE $2t = 1$ (NORMALIZATION)

AND THE SAME RESULT YIELDS

WHICH IMPLIES — DOES THIS SATISFY $\int \sigma_x \, dy = 0$ AT $x = \pm l$?

$$\sigma_x = 6AY \; ; \quad \sigma_y = \sigma_{xy} = 0$$

AND IN ORDER TO HAVE THE BOUNDARY CONDITIONS SATISFIED

$$A = -\frac{M}{4c^3}$$

$$\hookrightarrow \sigma_x = -\frac{3M}{2c^3} Y \; ; \quad \sigma_y = \sigma_{xy} = 0$$

NOW, COMING TO THE DISPLACEMENT FIELD

$$u_x = -\frac{3M}{2Ec^3} XY + f(Y)$$

$$u_y = \nu\frac{3M}{4Ec_3} Y^2 + g(x)$$

AND, AGAIN,

$$f(Y) = -\omega Y + u_0$$

$$g(x) = \frac{3M x^2}{4Ec^3} + \omega x + v_0$$

PLEASE NOTE THAT THE SAME PROBLEM IS TREATED IN THE CONTEXT OF SOLID MECHANICS ("STRENGTH OF MATERIALS") BY USING THE SO CALLED EULER-BERNOULLI BEAM THEORY. THERE THE NORMAL STRESSES ARE GIVEN BY

$$\sigma_x = -\frac{M}{I} y$$

WHERE I IS THE MOMENT OF INERTIA, WHICH CORRESPONDS TO $\frac{2c^3}{3}$ IF THE THICKNESS IS MADE EQUAL TO THE UNITY.

NOW, CONSIDERING AS BOUNDARY CONDITIONS FOR THE DISPLACEMENTS ARE $u_y(x=\pm l, 0) = 0$, $u_x(x=\pm l, 0)$ (WHICH PRECLUDES RIGID BODY MOTION) WE HAVE

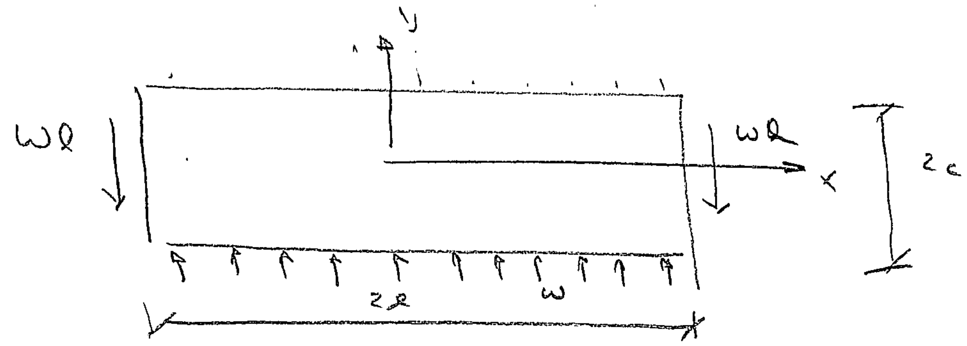
$$u_x = -\frac{Mxy}{EI} \quad \text{AND} \quad u_y = \frac{M}{2EI}\left(\nu y^2 + x^2 - l^2\right)$$

AND, FROM THE EULER-BERNOULLI THEORY

$$u_x(x, 0) = 0$$

$$u_y = \frac{M}{2EI}\left[x^2 - l^2\right]$$ (NOT CONSIDER THE POISSON EFFECT)

(3) BENDING OF A BEAM DUE TO A TRANSVERSE LOADING



WHERE w IS A UNIFORMLY DISTRIBUTED LOADING (FORCE/AREA).

BOUNDARY CONDITIONS:

$$\tau_{xy}(x, y = \pm c) = 0$$

$$\sigma_y(x, y = c) = 0$$

$$\sigma_y(x, y = -c) = +w$$

$$\int_{-c}^{c} \sigma_x(x = \pm\ell, y)\, dy = 0$$

$$\int_{-c}^{c} \sigma_x(x = \pm\ell, y)\, y\, dy = 0 \quad \text{(MOMENT AT EXTREMITIES)}$$

$$\int_{-c}^{c} \tau_{xy}(x = \pm\ell, y)\, dy = -w\ell$$

P.S.: NOTE THAT $\tau_{xy}(x = \pm\ell, y = \pm c) = 0$ (WHY?)

LET US TRY A POLYNOMIAL AS A TRIAL FOR THE AIRY STRESS FUNCTION:

$$\phi = A x^2 + Bx^2 Y + C Y^3 + D x^2 Y^3 + \frac{E}{5} Y^5$$

CAN YOU FIGURE OUT THE REASONS FOR PROPOSING THIS GUESS ?

WE START BY SATISFYING THE BOUNDARY CONDITIONS.

REMEMBER THAT

$$\sigma_{xy} = -\frac{\partial \phi}{\partial x \partial y} = -2BX + 6DXY^2$$

WHICH IS SYMMETRIC WITH RESPECT TO Y! MOREOVER IT CAN BE SET EQUAL ZERO REGARDLESS X FOR FIXED Y.

AS $\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = A + BY + DY^3$ WHICH IS NOT SYMMETRICAL WITH RESPECT TO Y (WHAT DOES IT SERVE TO?)

AND AS

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 6CY + 6DX^2Y + 4EY^3 \ldots$$ FOR A

FIXED X IS ANTI-SYMMETRICAL WITH RESPECT TO Y (WHY THIS SHOULD BE CONSIDERED?)

SO FAR WE HAVE MOTIVATED THE TERMS ASSOCIATED TO A, B, C AND D. WHAT ABOUT THE FIFTH-ORDER TERM?

THIS IS LEFT AS EXERCISE (HINT: REMEMBER THAT ϕ MUST SATISFY THE BI-HARMONIC EQUATION; YOU FIND ALSO THAT $E = -D$)

THEN, THE RESULTING STRESS-FIELD IS PROVIDED BY:

$$\sigma_x = 6CY + 6DX^2Y + 4EY^3 = 6CY + 6D\left(X^2Y - \frac{2}{3}Y^3\right)$$

$$\sigma_y = 2A + 2BY + 2DY^3$$

$$\sigma_{xy} = -2BX - 6DXY^2$$

NOW APPLYING THE FIRST THREE B.C. WE HAVE

$$A = -\frac{w}{4}\,; \quad B = \frac{3w}{8c} \quad \text{AND} \quad D = -\frac{w}{8c^3}$$

WITH THOSE RESULTS THE FOURTH AND SIXTH CONDITIONS ARE AUTOMATICALLY SATISFIED. NOW, ENFORCING THE FIFTH

$$C = -D\left(l^2 - \frac{2}{5}c^2\right) = \frac{w}{8c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)$$

NOW, LET US COMPARE THE STRESS FIELDS OBTAINED THROUGH ELASTICITY (PLANAR STRESS HYPOTHESIS) AND THE ONE COMING FROM THE EULER-BERNOUILLI THEORY)

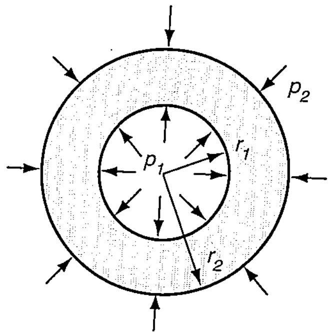
PLANE STRESS	EULER
$\sigma_x = \frac{w}{2I}(l^2 - x^2)y + \frac{w}{I}\left(\frac{y^3}{3} - \frac{c^2 y}{5}\right)$	$\sigma_x = \frac{My}{I} = \frac{w}{2I}(l^2 - x^2)y$
$\sigma_y = -\frac{w}{2I}\left[\frac{y^3}{3} - c^2 y + \frac{2}{3}c^3\right]$	$\sigma_y = 0$
$\tau_{xy} = -\frac{w}{2I}\,x\,(c^2 - y^2)$	$\tau_{xy} = \frac{VQ}{It} = \frac{-w}{2I}\,x\,(c^2 - y^2)$

COMPARING BOTH STRESS FIELDS, IMEDIATELLY WE RECOGNIZE THAT THE SHEAR COMPONENTS ARE IDENTICAL BUT THE NORMAL ONES ARE NOT. NOTE THAT THE MAXIMUM DIFERENCE BETWEEN THE TWO THEORIES OCCURS IN THE EXTREME FIBERS ($y = \pm c$) AND CORRESPONDS TO $\frac{w}{5}$ (REGARDLESS THE BEAM DIMENSIONS). ON THE OTHER HAND, COMPARING THE BOTH PARCELS OF σ_y WE CONCLUDE THAT FOR $l \gg c$ (TYPICAL WHEN DEALING WITH BEAMS AND ONE MAIN HYPOTHESIS OF THE EULER-BERNOUILLI THEORY) THE FIRST ONE TENDS TO BE BIGGER THAN THE SECOND, RENDERING THE RELATIVE DIFFERENCE BETWEEN THE TWO STRESS FIELDS SMALL.

COMPARING THE DIFFERENCES IN THE DISPLACEMENT FIELD IS LEFT AS AN EXERCISE.

(4) THICK-WALLED CYLINDER UNDER UNIFORM BOUNDARY PRESSURE

CONSIDERING A LONG HOLLOW CYLINDER UNDER AN INTERNAL AND EXTERNAL PRESSURES UNIFORM ALONG ITS LATERAL SURFACES CAN BE MODELED USING THE PLANE STRAIN HYPOTHESIS, WHICH IS SCHEMATICALLY PRESENTED IN THE FIGURE BELLOW.



NOW, WE SHOULD ALSO OBSERVE THAT GEOMETRY AND LOADING RESPECT AXISYMMETRY, AND THEY SHOULD BE DESCRIBED CONVENIENTLY BY ADOPTING POLAR COORDINATES r AND θ.

IN THAT CASE ALL INVOLVED QUANTITIES ARE ONLY FUNCTIONS OF r.

SO, IN WHAT CONCERNS THE AIRY FUNCTION $\phi(r)$

WE HAVE THE GOVERNING EQUATION (BI-HARMONIC EQUATION)

$$\nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \phi = 0$$

WITH

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \quad ; \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

AND $\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$

REDUCED TO

$$\sigma_{r\theta} = 0 \; ; \; \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \; ; \; \sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r}$$

AND

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} = \frac{1}{r}\frac{d}{dr} r\left(\frac{d}{dr}\right)$$

AND, CONSEQUENTLY

O.D.E 4th ORDER

$$\nabla^2\nabla^2\phi = \frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\phi = 0$$

LET US NOW CHECK ON THE BOUNDARY CONDITIONS FOR THE PROBLEM.

WE KNOW THAT

$$\sigma_{rr} = \frac{1}{r}\frac{d\phi}{dr} = -P_1 \qquad r = r_1 \text{ (INTERNAL RADIUS)}$$

$$\sigma_{rr} = \frac{1}{r}\frac{d\phi}{dr} = -P_2 \qquad r = r_2$$

DEPARTING FROM THESE RELATIONS THE DIFFERENTIAL INTEGRATION CAN BE INTEGRATED AND HAS AS FORMAL SOLUTION

$$\phi = A \ln r + B r^2 \ln r + C r^2 + D$$

THE CONSTANT D WILL BE IGNORED AS LONG AS IT REPRESENTS RIGID BODY MOTIONS THAT DOES NOT INTERFERE IN STRESS ANALYSIS. TO MAKE THE PROBLEM SOLVABLE B IS TAKEN EQUAL TO ZERO. THEREFORE THE SOLUTION IS

$$\phi = A \ln r + C r^2$$

$$\sigma_{rr} = \frac{1}{r}\frac{d\phi}{dr} = \frac{A}{r^2} + 2C \qquad A = \frac{r_1^2 r_2^2 (P_2 - P_1)}{r_2^2 - r_1^2}$$

$$\sigma_{\theta\theta} = \frac{d^2\phi}{dr^2} = -\frac{A}{r^2} + 2C \qquad C = \frac{1}{2}\,\frac{P_1 r_1^2 - P_2 r_2^2}{r_2^2 - r_1^2}$$

IN THE CASE OF NO EXTERNAL PRESSURE $P_2 = 0$ WE HAVE

$$\sigma_{rr} = \frac{r_1^2 P_1}{r_2^2 - r_1^2}\left(1 - \frac{r_2^2}{r^2}\right) < 0 \quad \text{(COMPRESSIVE STRESS)}$$

$$\sigma_{\theta\theta} = \frac{r_1^2 P_1}{r_2^2 - r_1^2}\left(1 + \frac{r_2^2}{r^2}\right) > 0 \quad \text{(TENILE STRESS)}$$

REMARK: THE "HOOP" STRESS AT THE BOUNDARIES:

$$r = r_2 \longrightarrow \sigma_{\theta\theta} = \frac{2 r_1^2}{r_2^2 - r_1^2} P_1$$

$$r = r_1 \longrightarrow \sigma_{\theta\theta} = \frac{r_1^2 + r_2^2}{r_2^2 - r_1^2} P_1 = SCF \, P_1$$

MAXIMUM STRESS

$$SCF = \frac{r_1^2 + r_2^2}{r_2^2 - r_1^2}$$ ... STRESS CONCENTRATION FACTOR (GEOMETRY)

REMARK: NOTE THAT IF THE WALL IS THIN ($r_2 = r_1 + t$; $t \ll 1$)

WE HAVE

$$\sigma_{\theta\theta} \simeq \frac{P_1 r_1}{t}$$ (THE SOLUTION PROVIDED IN THE SOL. MECH. COURSE)

IT IS ALWAYS WORTH TO REMEMBER THAT THE PLANE STRAIN MODEL CONTAINS AN OUT-OF-PLANE STRESS COMPONENT (IN THE PRESENT CASE THIS COMPONENT IS ALIGNED WITH THE LONGITUDINAL DIRECTION OF THE CYLINDER)

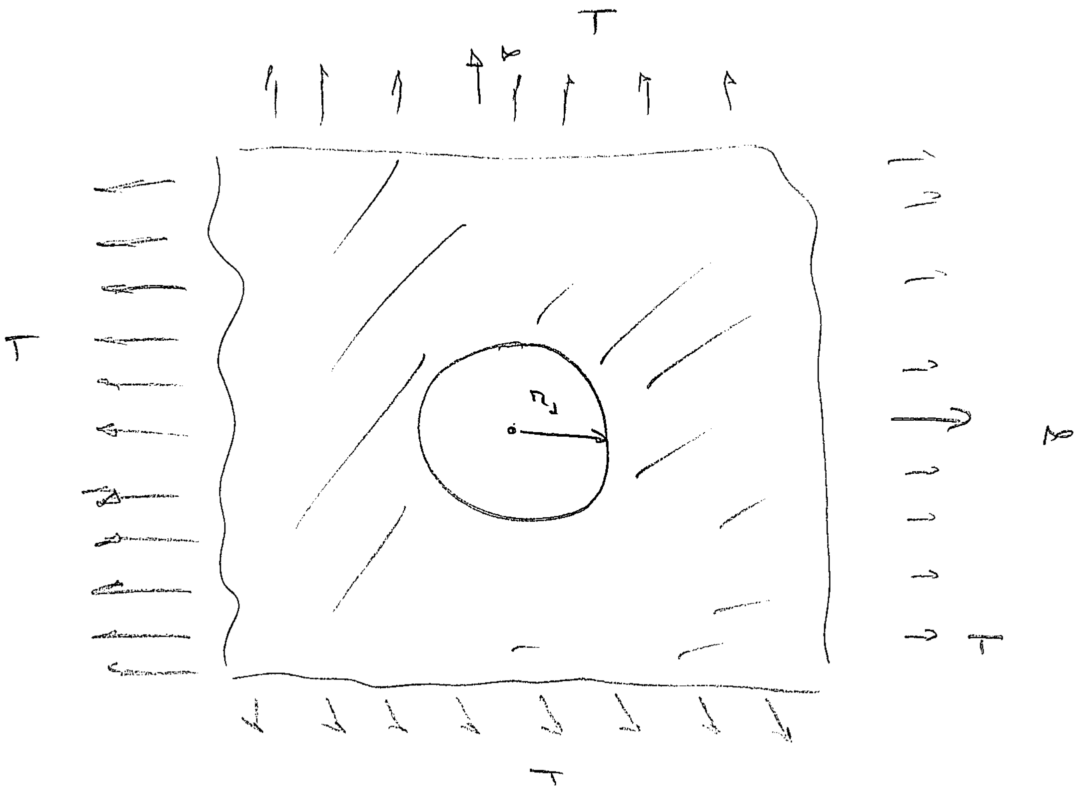
$$\sigma_z = \nu(\sigma_r + \sigma_\theta) = 2\nu \frac{r_1^2 p_1^2 - r_2^2 p_2^2}{r_2^2 - r_1^2}$$

THE RADIAL DISPLACEMENT IS GIVEN BY

$$u_r = \frac{1+\nu}{E}\left[-\frac{r_1^2 r_2^2 (p_2 - p_1)}{r_2^2 - r_1^2}\frac{1}{r} + (1-2\nu)\frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2} r\right]$$

THE SOLUTION OBTAINED ABOVE CAN BE CONVENIENTLY EXPLORED TO GENERATE, THROUGH APPROPRIATE LIMITING PROCESS, THE SOLUTION OF OTHER PROBLEMS AS THE FORTHCOMING EXAMPLES.

(1) STRESS - FREE HOLE IN AN INFINITE MEDIUM UNDER LOADING AT INFINITY



THE SOLUTION OF THE PRESENT PROBLEM CAN BE OBTAINED BY TAKING THE PREVIOUS EXAMPLE WITH $r_2 \to \infty$, $P_1 = 0$ AND $P_2 = -T$ (WHERE T CORRESPONDS TO AN EQUAL BIAXIAL LOADING AT INFINITY). UNDER THESE CONDITIONS

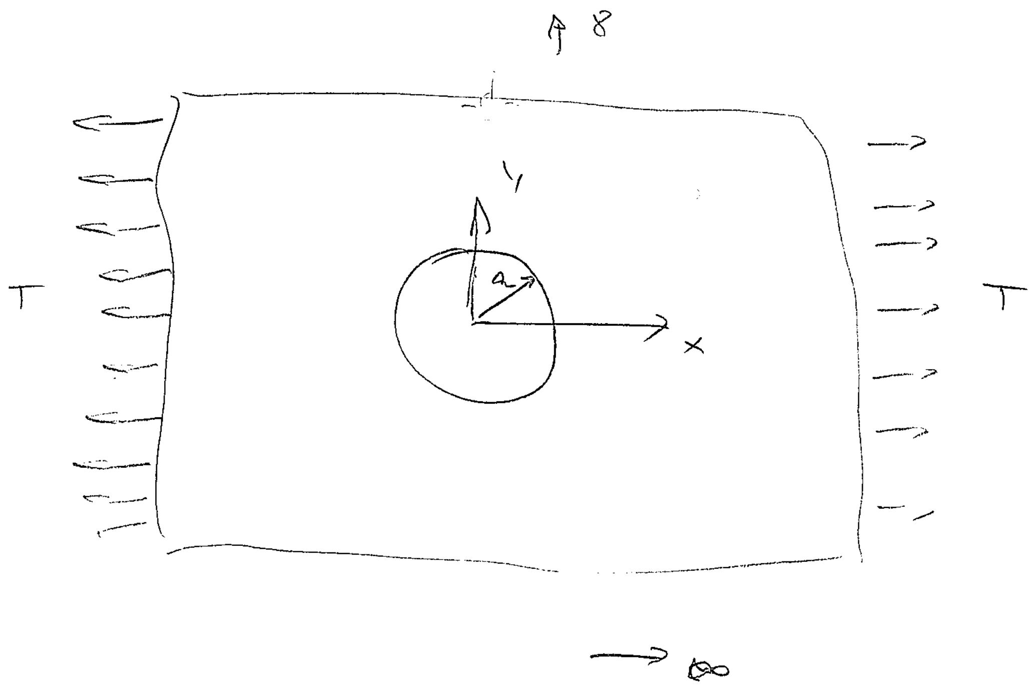
$$\sigma_r = T\left(1 - \frac{r_1^2}{r^2}\right) \quad \text{AND} \quad \sigma_\theta = T\left(1 + \frac{r_1^2}{r^2}\right)$$

AND THE MAXIMUM STRESSES OCCUR AT $r = r_1$

$$\sigma_{MAX} = (\sigma_\theta)_{MAX} = 2T$$

(5) Infinite Medium with a Stress-Free Hole under Uniform Far-Field Tension Loading

This example strongly differs from the previous one as the axisymmetry is lost due to the loading. This implies that solution is also θ-dependent.



Boundary Conditions

$$\sigma_r(a,\theta) = \sigma_{r\theta}(a,\theta) = 0 \quad \text{(internal pressure} \searrow 0)$$

$$\sigma_r(r,\theta) = \frac{T}{2}(1+\cos 2\theta) \qquad r \nearrow \infty$$

$$\sigma_\theta(r,\theta) = \frac{T}{2}(1-\cos 2\theta) \qquad r \nearrow \infty$$

$$\sigma_{r\theta}(r,\theta) = -\frac{T}{2}\,\text{sen}\, 2\theta \qquad r \nearrow \infty$$

REMARK: THE ABOVE BOUNDARY CONDITIONS CAN BE OBTAINED BY THE USE OF THE STRESS TRANSFORMATION RELATIONS (HOW? THIS IS A GOOD EXCERCISE! START BY CHECKING THE OBVIOUS DIRECTIONS $\theta = 0$ AND $\theta = \frac{\pi}{2}$)

THE SOLUTION OF THE PROBLEM USES A SUPERPOSITION VIEW OF THE SITUATION. THUS, LET US DETERMINE FIRST THE STRESS STATE IN THE MEDIUM WITH NO HOLE. THIS IS SIMPLY $\sigma_x = T$ AND $\sigma_y = \sigma_{xy} = 0$ (THIS IS NOTHING BUT AN EXTENSION OF EXAMPLE 1 AND, AGAIN, CAN BE DERIVED FROM THE AIRY FUNCTION $\phi = \frac{1}{2} T Y^2 = \frac{T}{2} r^2 \sin^2\theta = \frac{T}{4} r^2 (1 - \cos 2\theta)$).

CONSIDERING THAT THE PRESENCE OF THE HOLE AS A DISTURBANCE IN THIS HOMOGENEOUS FIELD, AND THUS ITS EFFECTS SHOULD DECAY TO ZERO FOR $r \gg a$, THE FOLLOWING AIRY FUNCTION IS PROPOSED

$$\phi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r +$$
$$+ (a_{21} r^2 + a_{22} r^4 + a_{23} r^{-2} + a_{24}) \cos 2\theta$$

INDEED, THIS TRIAL FUNCTION COMES FROM A SELECTION OF FEW TERMS OF THE SO CALLED MICHELL SOLUTION OF THE BI-HARMONIC EQUATION THROUGH THE USE OF FOURIER SERIES. THIS SOLUTION IS GIVEN BY THE FOLLOWING EXPRESSION

$$\begin{aligned}
\phi = {} & a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r + \\
& + (a_4 + a_5 \log r + a_6 r^2 + a_7 r^2 \log r)\theta \\
& + \left(a_{11} r + a_{12} r \log r + \frac{a_{13}}{r} + a_{14} r^3 + a_{15} r\theta + a_{16} r\theta \log r\right) \cos\theta \\
& + \left(b_{11} r + b_{12} r \log r + \frac{b_{13}}{r} + b_{14} r^3 + b_{15} r\theta + b_{16} r\theta \log r\right) \operatorname{sen}\theta \\
& + \sum_{m=2}^{\infty} \left(a_{m_1} r^m + a_{m_2} r^{2+m} + a_{m_3} r^{-m} + a_{m_4} r^{2-m}\right) \cos m\theta \\
& + \sum_{m=2}^{\infty} \left(b_{m_1} r^m + b_{m_2} r^{2+m} + b_{m_3} r^{-m} + b_{m_4} r^{2-m}\right) \operatorname{sen} m\theta
\end{aligned}$$

THIS SHOULD BE LOOKED AT AS A FORMAL (OR REFERENCE) SOLUTION DUE TO THE INFINITY SERIES IT CONTAINS.

NOTE THAT THE FOUR FIRST TERMS CORRESPONDS

TO AN AXISYMMETRIC RESPONSE AND

$$\nabla^2\nabla^2 \phi_{AX} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2 \phi_{AX} =$$

$$\nabla^2 \left(-a_1 \cancel{r^{-2}} + 2a_2 + a_3(2\ln r + 3) + a_1 \cancel{r^{-2}} + 2a_2 + a_3(2\log r + 1) \right) =$$

$$= -a_3 2 r^{-2} - a_3 2 r^{-2} + a_3 2 r^{-2} + a_3 2 r^{-2} = 0 .$$

THEN IT IS ALSO A SOLUTION OF THE BI-HARMONIC AND CAN BE USED AS A GUESS FOR THE SOLUTION OF ANY AXISYMMETRICAL PROBLEM.

THEREFORE THE PROPOSED GUESS FOR THE SOLUTION OF THE PRESENT COMPONENT IS INSPIRED IN THE ASYMPTOTIC BEHAVIORS:

$r \searrow a$ THE SOLUTION KEEPS AN AXISYMMETRIC CHARACTER

$r \nearrow \infty$ THE HOLE EFFECT IS ONLY A PERTUBATION OF THE UNIFORM FIELD ($\cos 2\theta$ TERM)

REMARK: YOU SHOULD VERIFY IF IT SATISFIES THE BI-HARMONIC EQUATION..

THE STRESSES CORRESPONDING TO THE PROPOSED AIRY FUNCTION ARE:

$$\sigma_r = a_3(1 + 2\log r) + 2a_2 + \frac{a_1}{r^2} - \left(2a_{21} + 6\frac{a_{23}}{r^4} + \frac{4a_{24}}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = a_3(3 + 2\log r) + 2a_2 - \frac{a_1}{r^2} + \left(2a_{21} + 12a_{22}r^4 + \frac{6a_{23}}{r^4}\right)\cos 2\theta$$

$$\sigma_{r\theta} = \left(2a_{21} + 6a_{22}r^2 - 6\frac{a_{23}}{r^4} - 2\frac{a_{24}}{r^2}\right)\sin 2\theta$$

SO WE HAVE TO FIND THE CONSTANTS a_{ij}. THE FIRST THING TO BE REALIZED CONCERNS a_3 AND a_{22}. IF THEY ARE TAKEN DIFFERENT FROM ZERO THE STRESSES WILL BE NOT FINITE AS $r \nearrow \infty$, THEN $a_3 = a_{22} = 0$.

NOW, APPLYING THE BOUNDARY CONDITIONS

$$2a_2 + \frac{a_1}{a_2^2} = 0$$

$$2a_{21} + \frac{6a_{23}}{a^4} + \frac{4a_{24}}{a^2} = 0$$

$$2a_{21} - \frac{6a_{23}}{a^4} - \frac{2a_{24}}{a^2} = 0$$

$$2a_{21} = -\frac{T}{2}$$

$$2a_2 = T_2$$

AND THEN

$$a_1 = -\frac{a^2T}{2}, \quad a_2 = \frac{T}{4}, \quad a_{21} = -\frac{T}{4}, \quad a_{23} = -\frac{a^4T}{4}, \quad a_{24} = \frac{a^2T}{2}$$

LEADING TO

$$\sigma_r = \frac{T}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{T}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{T}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{T}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_{r\theta} = -\frac{T}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

THE HOOP STRESS AROUND THE HOLE IS GIVEN BY

$$\sigma_\theta(a,\theta) = T(1-2\cos 2\theta)$$

WHICH VANISHES AT $\theta = 30^\circ$ AND LEADS TO A MAXIMUM AT $\theta = 90^\circ$ (ONLY LOOKING AT $0<\theta<90^\circ$). INDEED THIS MAXIMUM IS

$$\sigma_\theta(a,\theta)\Big|_{max} = 3\,T$$

STRESS CONCENTRATION (WHICH IS GREATER THAN THE PREVIOUS EXAMPLE)

AT THA POINT YOU MAY WANT TO FOLLOW A DIFFERENT PATH (MAYBE CLEVER!) FOR SOLVING THE SAME PROBLEM.

SO LET'S REMEMEMBER THAT WE'VE SOLVED A SIMILAR SITUATION BEFORE IN WHICH WE HAD TRACTIONS IN THE INFINITY IN BOTH DIRECTIONS. SO, WHY NOT EXPLORE THE SUPERPOSITION PRINCIPLE?

NOTE THAT THE ACTUAL BOUNDARY CONDITIONS (THAT PLAY THE ROLE OF LOADS IN THE PRESENT SITUATION) CAN BE DECOMPOSED INTO (WE WILL JUST DEAL WITH THOSE AT $r \to \infty$)

$$\sigma_r(r,\theta) = \frac{T}{2} + \frac{T}{2}\cos 2\theta$$

$$\sigma_\theta(r,\theta) = \frac{T}{2} - \frac{T}{2}\cos 2\theta$$

$$\sigma_{r\theta}(r,\theta) = \underbrace{\phantom{\frac{T}{2}}}_{\text{PART I}} \quad \underbrace{-\frac{T}{2}\sin 2\theta}_{\text{PART II}}$$

THEREFORE WE CAN SOLVE OUR ORIGINAL P AS THE DIRECT SUMS OF P_1 AND P_2, CORRESPONDING TO BOUNDARY CONDITIONS LABELLED AS PART I AND PART II ABOVE. FOR BOT P_1 AND P_2, AT $r = a$ 0 BOUNDARY CONDITIONS ARE ASSUMED.

PLEASE NOTE THAT P_1 IS NOTHING BUT EXAMPLE (4). SO IT REMAINS TO SOLVE P_2 (WHICH SEEMS TO BE AT LEAST SLIGHTLY SIMPLER THAN THE ORIGINAL PROBLEM, DO YOU AGREE?)

THE BOUNDARY CONDITIONS OF P_2 STRONGLY SUGGESTS

$\phi = f(r) \cos 2\theta$ (REMEMBER THAT SECOND DERIVATIVES OF COS OR SIN COINCIDE (UP TO A SIGNAL) TO THE ORIGINAL FUNCTIONS).

AS ϕ MUST SATISFY THE BI-HARMONIC EQUATION, WHICH IMPLIES

O. D. E. !

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right) f(r) = 0$$

SO WE CAN TRY $f(r) = r^\lambda$

$$\hookrightarrow \lambda = 0, 2, -2, 4$$

$$\hookrightarrow f(r) = C_1 r^2 + C_2 r^4 + C_3 r^{-2} + C_4$$

THEN, USING THE B.C. OF P_2

$$C_1 = -\frac{1}{4} T \ , \ C_2 = 0 \ , \ C_3 = -\frac{1}{4} a^4 T \ \text{AND} \ C_4 = \frac{a^2 T}{2}$$

AND WE RETRIEVE THE SOLUTION OBTAINED PREVIOUSLY.

JUST REMEMBER