# CHAPTER 4

## MATERIAL BEHAVIOR -

## LINEAR ELASTIC CONSTITUTIVE EQUATION

JUST AS A SUMMARY, SO FAR WE HAVE OBTAINED THE FOLLOWING RELATIONS INVOLVING STRAINS AND STRESSES:

$$\varepsilon(\underline{u}) = \frac{1}{2}\left(D\underline{u} + D\underline{u}^T\right)$$

$$\operatorname{div} T + \underline{b} = \underline{0}$$

$$T = T^T$$

THERE ARE 15 UNKNOWNS ($\varepsilon_{ij}$, $u_i$, $T_{ij}$) AND 9 EQUATIONS (CONSIDERING THE SCALAR EQUATIONS CORRESPONDING TO WRITING THE FIRST TWO RELATIONS ABOVE IN COMPONENTS AND CONSIDERING THE THIRD ONE). SO, THERE ARE STILL 6 RELATIONS MISSING. THEY WILL BE INTRODUCED IN THIS CHAPTER AND THEY CORRESPOND TO THE ELASTIC RESPONSE OF THE MATERIALS. THROUGHOUT THE REST OF THESE NOTES, THEY WILL BE REFERRED TO AS CONSTITUTIVE EQUATIONS.

# 4.1) Material Behavior - Elastic Constitutive Equations

# Constitutive Equations are intend to describe the material response through a set of relations involving stress and strain tensors. They are built combining the theoretical background with experiments.

# The development of constitutive equations should not violate any laws of thermodynamics, but it is mainly focused on complying with experimental measurements. Besides, at least ideally, it can be guided by some underlying physical mechanism that governs the response of the solid.

Remark: Constitutive Equations here conform to continuum theories that are built to model the behavior of structural components, with dimensions tipically ranging from 0.1 to 100 m. So, this length scale is under direct control

THE LINEAR ELASTIC CONSTITUTIVE EQUATIONS, ALTHOUGH NOT TAKING INTO ACCOUNT DIRECTLY THE MICRO SCALE IN WHICH VERY COMPLEX PHYSICAL PHENOMENA TAKE PLACE, CAN OFTEN BE USED WHICH SUCCESS TO DESCRIBE THE MATERIAL RESPONSE. ON THE OTHER HAND, WHEN DEALING WITH MODELLING NON-ELASTIC BEHAVIOR (e.g PLASTICITY, FRACTURE MECHANICS, VISCO-ELASTICITY) INCORPORATING THE FINE SCALES MECHANISMS HAS LEAD TO SIGNIFICANT IMPROVEMENTS. BECAUSE OF THAT AND ALLIGNED WITH THE THRIVING WORLD OF NANO-MECHANICS, THE MULTISCALE MODELING FIELD HAS EXPERIENCED, IN THE LAST FEW YEARS, AN ENOURMOUS BOOSTING. THOSE COMMENTS ARE, MAINLY, TO MAKE YOU AWARE OF THE LIMITATIONS ON THE CONSTITUTIVE THEORY YOU ARE ABOUT TO BE INTRODUCED.

LINEAR ELASTIC CONSTITUTIVE LAWS ARE PRONE TO MODEL ALMOST SOLID MATERIALS UNDER SUFFICIENTLY SMALL LOADS (WHICH USUALY IMPLIES THAT THEY ARE UNDERGOING SMALL DEFORMATIONS), INCLUDING CERAMICS, METALS AND MOST POLYMERS. UNDER THESE CONDITIONS, THE EXPERIMENTAL OBSERVATIONS CONDUCT TO THE FOLLOWING CHARACTERIZATION:

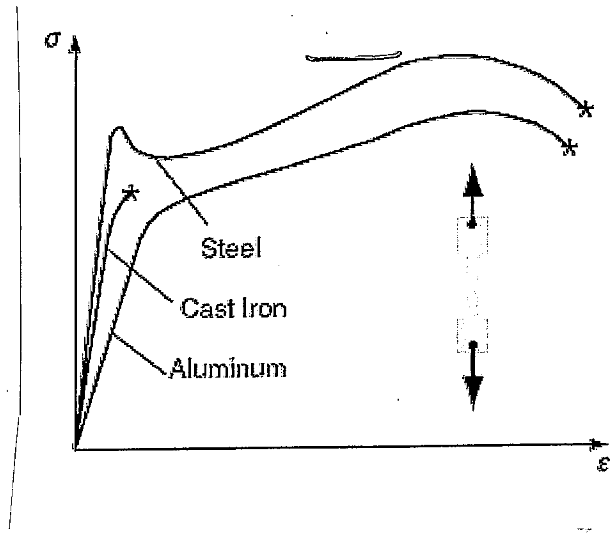
- THE PROCESS (ASSOCIATED TO THE MOTION) IS PERFECTLY REVERSIBLE, WHICH MEANS THAT (KEEPING THE TEMPERATURE CONSTANT IN THE NEAR ENVIRONMENT) IF THE BODY IS SUBMITTED TO A CLOSED CYCLE OF LOADS THE NET WORK DONE IN THE SOLID IS ZERO;

- THE STRESS AT A POINT DEPENDS UNIQUELY ON SOME APPROPRIATE MEASURE OF STRAIN AT THAT POINT AND TEMPERATURE, AND IS INDEPENDENT OF HISTORY OR LOADING RATE;

- STRESS AND STRAIN ARE RELATED BY A LINEAR FUNCTION;

- STRAINS ARE SMALL ENOUGH TO BE ACCURATELY MEASURED BY THE LINEARIZED STRAIN TENSOR

## - Basic Experiment: Tension Test



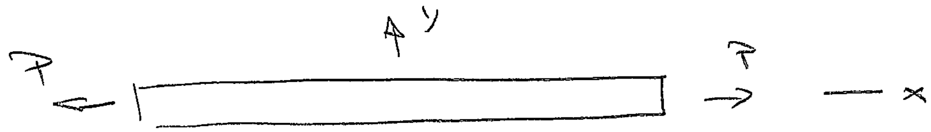

Initially the stress-strain relation is linear, which changes when the load achieves a certain threshold and a nonlinear behavior, comprehending permanent and large deformations, is observed untill the specimen ends up breaking (failure). This stress threshold is often referred to as the proportional limit, from which once load is released the sample returns to its original shapes ($\varepsilon_f = 0$). Indeed proportional limit, elastic limt or yield point are almost the same.

THUS, IN THE LINEAR ELASTIC INTERVAL

$$\sigma = E \, \varepsilon$$

YOUNG MODULUS

AND, MOREOVER, SMALL CONTRACTION ON THE TRANSVERSE DIRECTION IS ALSO OBSERVED




$$\sigma_x = \frac{P}{A}$$ ("ENGINEERING STRAIN")

$$\sigma_x = E \, \varepsilon_x$$

AND

$$\varepsilon_y = \varepsilon_z = -\nu \, \varepsilon_x$$

POISON RATIO

THIS LINEARITY OBSERVED IN THE TENSION TEST CAN BE USED TO HELP ON THE CONSTRUCTION OF GENERAL 3-D LINEAR CONSTITUTIVE EQUATION GIVEN BY:

$$T = C\,\varepsilon$$

WHERE $C$ IS A FOURTH-ORDER TENSOR, WHICH IS EXPRESSED IN COMPONENTS (81!) IS

$$T_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

AND, BY CONSTRUCTION, $C$ IS SYMMETRIC:

$$C_{ijkl} = C_{jikl}$$

AND (SO CALLED MINOR SYMMETRY)

$$C_{ijkl} = C_{ijlk}$$

↳ REDUCING THE TOTAL NUMBER OF PARAMETERS FROM 81 TO 36.

INDEED THE RELATION CAN BE CAST IN THE FOLLOWING MATRICIAL FORM

$$\underbrace{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots \\ \vdots & \ddots & & \vdots \\ & & \ddots & \vdots \\ \vdots & & & \ddots \end{bmatrix}}_{\text{WHICH MUST BE SYMMETRIC (MAJOR SYMMETRY)}} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{bmatrix}$$

THUS THE 36 INDEPENDENT ARE FURTHER REDUCED TO 21.

THE $C_{ij}$ ARE CALLED ELASTIC MODULI AND HAVE UNITS OF STRESS (FORCE / AREA). AND USUALLY ASSUMED HAVING CONSTANT VALUES ACROSS THE MATERIAL (HOMOGENEITY).

THE ELASTIC THEORY PRESENTED HERE ASSUMES ISOTROPY

WHICH IS TO SAY THAT THERE ARE NO PARTICULAR DIRECTIONS IN THE RESPONSE OF THE MATERIAL (INDEED, SOME MATERIALS BEHAVE DIFFERENTLY, WHEN THEI CONSTITUTIVE RESPONSE VARY IN ALL DIRECTIONS THE MATERIAL IS REFFERED TO AS ANISOTROPIC). ISOTROPY IMPLIES THE $C$ CAN BE WRITTEN AS

$$C_{ijkl} = c_1 \delta_{ij} \delta_{kl} + c_2 \delta_{ik} \delta_{jl} + c_3 \delta_{il} \delta_{jk}$$

BY IMPOSING THE SYMMETRY CONDITIONS INTRODUCED BEFORE, $C$ CAN BE EXPRESSED IN TERMS OF JUST TWO CONSTANTS

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$

ONLY TWO PARAMETERS.
(LAMÉ CONSTANTS)

NOTE THAT TWO PARAMETERS CAN BE IDENTIFIED THROUGH THE TENSION TEST, NAMELY: $E$ AND $\nu$

$$E = \frac{\sigma_x}{\varepsilon_x} \qquad \text{AND} \qquad \nu = -\frac{\varepsilon_y}{\varepsilon_x}$$

AND

$$\varepsilon_{ij} = \underbrace{S_{ijkl}}_{\substack{\text{THE INVERSE OF} \\ C \\ \text{(SOMETIMES CALLED} \\ \text{COMPLIANCE TENSOR)}}} \sigma_{kl} = \left(d_1 \delta_{ij}\delta_{kl} + d_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right)\sigma_{kl}$$

AND

$$\varepsilon_{ij} = d_1 \overbrace{\sigma_{kk}}^{\{\sigma_{xx}+\sigma_{yy}+\sigma_{zz}\}} \delta_{ij} + 2 d_2 \sigma_{ij}$$

AS IN THE TENSION TEST $\sigma_{xx} \neq 0$ BUT $\sigma_{yy} = \sigma_{zz} = 0$

IN THIS CASE

$$\varepsilon_{xx} = d_1 \sigma_{xx} + 2 d_2 \sigma_{xx} = \frac{\sigma_{xx}}{E}$$

AND

$$\varepsilon_{yy} = d_1 \sigma_{11} = -\nu\, \varepsilon_{xx} = -\nu \frac{\sigma_{xx}}{E}$$

THEN

$$d_1 = \frac{-\nu}{E} \qquad \text{AND} \qquad d_2 = \frac{1+\nu}{2E}$$

$$S_{ijkl} = \frac{-\nu}{E} \delta_{ij}\delta_{kl} + \frac{1+\nu}{2E}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

SO WE OBTAIN THE HOOKE'S LAW:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

AND

$$\varepsilon_{qq} = \frac{1+\nu}{E}\sigma_{qq} - \frac{3\nu}{E}\sigma_{kk} = \frac{1-2\nu}{E}\sigma_{qq}$$

AND REMEMBER

$$\underbrace{\frac{\Delta V}{V}}_{\text{RATIO OF VOLUME CHANGE}} = \varepsilon_{qq} = \frac{1-2\nu}{E}\sigma_{qq} = \underbrace{\frac{3(1-2\nu)}{E}}_{\text{BULK MODULUS}} p$$

INVERTING HOOKE'S LAW:

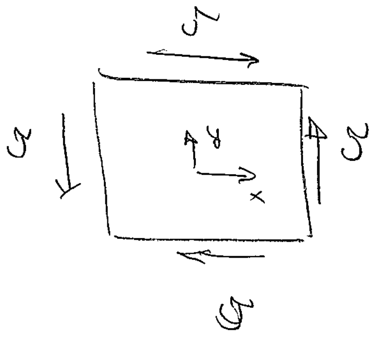
$$\frac{1+\nu}{E}\sigma_{ij} = \varepsilon_{ij} + \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\Longrightarrow \sigma_{ij} = \frac{E}{2(1+\nu)}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$$

So $$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

NOW CONSIDERING A PURE SHEAR




$$G = \frac{\tau}{\gamma}$$

↪ ANG. DIST.

$$\tau_{xy} = 2\mu\,\varepsilon_{xy} \Longrightarrow G = \mu = \frac{E}{2(1+\nu)}$$

THE HOOKE'S LAW CAN BE GENERALIZED TO TAKE INTO ACCOUNT THERMAL EFFECTS BY EXPLORING THE SUPERPOSITION PRINCIPLE YIELDING

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\,\Delta\theta\,\delta_{ij}$$

WHERE $\alpha$ IS THE COEFFICIENT OF THERMAL EXPANSION AND $\Delta\theta$ STANDS FOR THE TEMPERATURE DIFFERENCE BETWEEN THE ACTUAL TEMPERATURE $\theta$ AND THE REFERENCE ONE $\theta_0$.

REMARK: $-1 < \nu < 1/2$, OTHERWISE WE WOULD HAVE $\Delta V \lesssim 0$