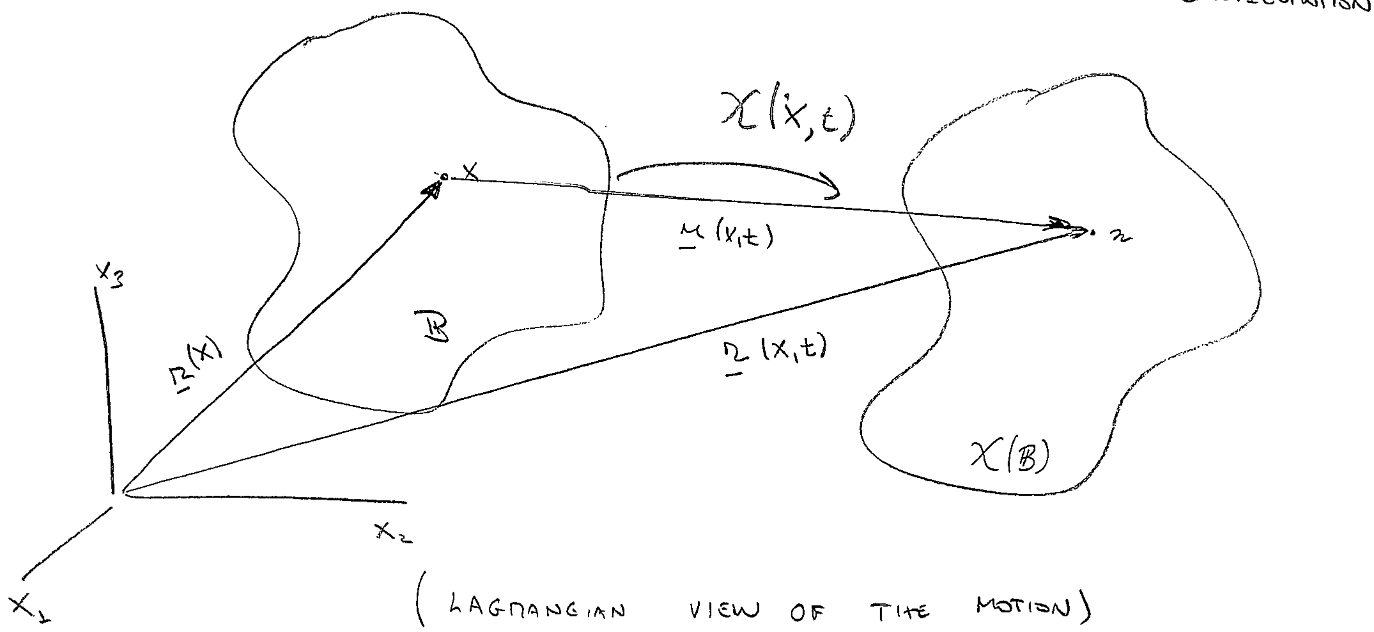
CHAPTER 2

KINEMATICS: DEFORMATION AND STRAINS

2.1) DEFORMATION AND STRAIN



(LAGRANGIAN VIEW OF THE MOTION)

• WE BEGIN THE DEVELOPMENT OF BASIC FIED EQUATIONS OF ELASTICITY BY INVESTIGATING THE KINEMATICS (GEOMETRY) OF BODIES DEFORMATION. AS A MATTER OF FACT, THIS DEFORMATION RESULTS FROM THE APPLICATON OF LOADS OVER THE BODY. SO, EXTERNAL FORCES, DEFORMATION AND MATERIAL BEHAVIOR ARE THE 3 MAJOR ASPECTS WHEN ONE INTENDS TO REALLY UNDERSTAND THE MECHANICS OF A BODY UNDERGOING DEFORMATION. ONLY FOR DIDACTICAL PURPOSES THEY ARE PRESENTED IN DIFFERENT CHAPTERS.

THE MOTION OF A DEFORMABLE BODY IS FORMALLY DESCRIBED BY THE MAPPING $\mathcal{X}(x,t)$ WHICH MAPS $\mathcal{B}$ (THE REFERENCE CONFIGURATION CAN BE ARBITRARILY CHOOSEN, BUT OFTEN IT IS ADOPED AS A NON DEFORMED CONFIGURATION), AT EACH INSTANT t, ONTO A CLOSED REGION OF THE SPACE WHICH IS BEING OCCUPIED BY THE BODY. IT IS IMPORTANT TO NOTICE THAT THE INTTRINSIC GEOMETRY OF THE BODY MIGHT HAVE CHANGED, THEREFORE, FOR A FIXED MOMENT t^*, $\mathcal{X}(x, t^*)$ DESCRIBES DE DEFORMATION.

THE VECTOR FIELD

$$\underline{u}(x,t) = \mathcal{X}(x,t) - X$$

REPRESENTS THE DISPLACEMENT OF X.

<u>REMARKS</u> (i) DESCRIBING THE MOTION THROUGH $\mathcal{X}$ ENTAILS A "LAGRANGIAN VIEW" AS FIXING A POINT IN $\mathcal{B}$, $\mathcal{X}$ PROVIDES THE MOTION OF THIS POINT (OFTEN REFFERED TO AS A PARTICLE DUE THE SIMILARITY OF THE SITUATION WITH BASIC PHYSIC) ALONG THE TIME

(ii) WE HAVE BEEN DEALING WITH POINTS AND VECTORS AS DIFFERENT MATHEMATICAL OBJECTS, BUT THEY CAN BE EASILY IDENTIFIED BY REGARDING A POSITION VECTOR $\underline{R}$ (JUST AS DEPCITED IN THE FIGURE OF PAGE 2.1)

(iii) INDEED, THE MOTION CAN TAKE PLACE AND NO DEFORMATION OCCURS, WHICH IS REFFERED TO AS RIGID BODY MOTION

THE LAST REMARK SUGGESTS THAT χ ITSELF MIGHT NOT BE A GOOD WAY OF CHARACTERIZING THE DEFORMATION. INDEED, LET US FOLLOW SOME INTUITIVE LINES. IMAGINE A SIMPLE EXPERIMENT IN WHICH A THIN RUBBER MENBRANE IS PUSHED BY ITS BORDERS. IF BEFORE PUSHING THE MEMBRANE A RECTANGULAR GRID WAS PAINTED ON ITS SURFACE, A COMPLETED DISTORDED MESH WOULD BE SEEN AFTER DEFORMATION

THIS SIMPLE OBSERVATION RECALL THE FACT THAT DEFORMATION IS LOCAL AND REFLECTS RELATIVE CHANGES ACROSS THE NEIGHBORHOOD.

THUS, A GOOD DEFORMATION MEASURE IS BASED ON DERIVATIVES, AS THOSE EXPRESS LOCAL VARIATIONS.

LET US NOW TAKE THE GRADIENT TENSOR

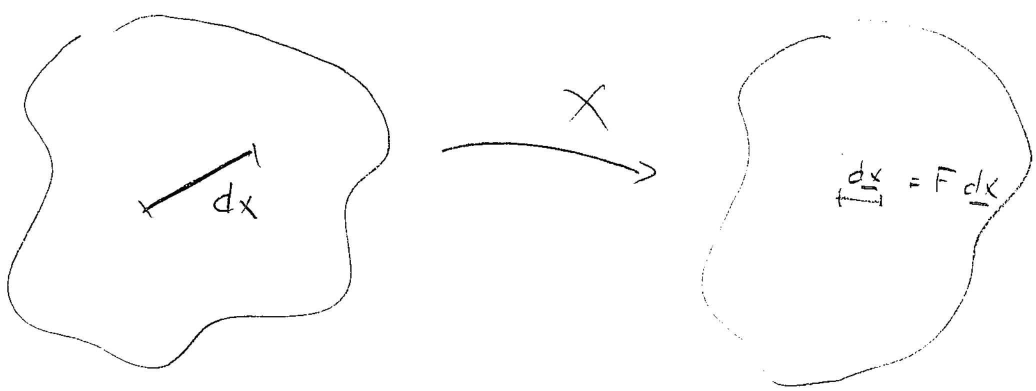
$$F = \nabla x$$

PLEASE NOTE THAT IF $\underline{dX}$ DENOTES AN INFINITESIMAL VECTOR ASSOCIATE TO A POINT X IN THE REFERENCE CONFIGURATION, THEN

$$\underline{dx} = F\,\underline{dX}$$

WHERE $\underline{dx}$ WOULD BE THE VECTOR $\underline{dX}$ AFTER DEFORMATION.

(THINK ON $\underline{dX}$ COINCIDING WITH ONE EDGE OF THE GRID)



THEN

$$\underline{dx} \cdot \underline{dx} = F\,\underline{dX} \cdot F\,\underline{dX} = \underline{dX} \cdot F^T F\,\underline{dX}$$

$$\|\underline{dx}\|^2 = \underline{dX} \cdot F^T F\,\underline{dX}$$

WHICH MEANS THAT, SOMEHOW F, AT LEAST, EXPRESSES LOCAL CHANGES ON LENGHTS. LET US GO FURTHER, ADMIT THAT $\underline{dX}$ CONNECT TWO NEIGHBORING POINTS IN THE UNDEFORMED CONFIGURATION. THEN, $\underline{dx}$ LINKS THE SAME "PARTICLES" (REMEMBER THE LAGRANGIAN DESCRIPTION) IN THE DEFORMED CONFIGURATION.

THUS

$$\Delta = \|\underline{dx}\|^2 - \|\underline{dX}\|^2 = \underline{dx} \cdot \underline{dx} - \underline{dX} \cdot \underline{dX} =$$

$$= F\,\underline{dX} \cdot F\,\underline{dX} - \underline{dX} \cdot \underline{dX} =$$

$$= \underline{dX} \cdot F^T F\,\underline{dX} - \underline{dX} \cdot \underline{dX} =$$

$$= \underline{dX} \cdot (F^T F - I)\,\underline{dX}$$

AND $E = \frac{1}{2}(F^T F - I)$ IS A MEASURE OF THE DEFORMATION IN THE SENSE IT PROVIDES A MEASURE OF THE LENGHT CHANGE. THE TENSOR E IS REFFERED TO AS THE LAGRANGEAN STRAIN TENSOR (OR GREEN'S STRAIN TENSOR

REMARK: SOMETIMES IT PROVES TO BE USEFULL DEVELOPING AN EQUIVALENT DEFORMATION MEASURE TO BE APPLIED IN THE DEFORMED CONFIGURATION. IT IS OBTAINED BY CONSIDERING (AS $\underline{dX} = F^{-1}\underline{dx}$)

$$\Delta = \underline{dx} \cdot \underline{dx} - F^{-1}\underline{dx} \cdot F^{-1}\underline{dx} = \underline{dx} \cdot \underline{dx} - \underline{dx} \cdot F^{-T}F^{-1}\underline{dx}$$

$$\Delta = \underline{dx} \cdot (I - F^{-T}F^{-1})\underline{dx}$$

LEADING TO $E^{*} = \frac{1}{2}\left(I - F^{-T}F^{-1}\right)$ CONSIDERED THE EULERIAN STRAIN TENSOR.

NOW RECAL THE POLAR DECOMPOSITION THEOREM (INTRODUCED IN THE PREVIOUS CHAPTER) WHICH SAYS THAT ANY TENSOR WITH A POSITIVE DETERMINANT MAY BE DECOMPOSED THROUGH THE PRODUCT OF A SYM. POS. DEF TENSOR AND A ROTATION. IT CAN BE THUS APPLIED TO F (WE WILL SEE LATER ON WHY $\det F > 0$) LEADING TO

$$F = RU = VR$$

AND

$$U^2 = F^T F \quad \text{AND} \quad V^2 = FF^T$$

$$(\text{REMEMBER } RR^T = R^T R = I)$$

LET US NOW EXPLORE THIS LAST RESULT. FOR THAT, CONSIDER A SIMPLE DEFORMATION GIVEN BY A STRETCH WITH A FIXED POINT X_0, WHICH IS

$$\chi(x) = X_0 + \underline{\underline{A}}\,(x - X_0)$$

WITH $\underline{\underline{A}}$ SYMMETRIC AND POSITIVE DEFINITE AND IF THIS STRETCH IS ONLY IN $\underline{e}_1$ DIRECTION, $\underline{\underline{A}}$ IS GIVEN BY

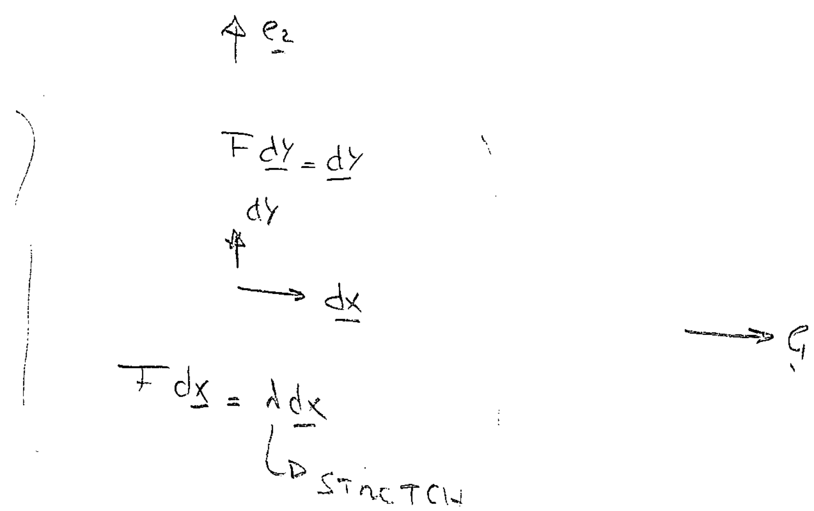
$$\underline{\underline{A}} = \underline{\underline{I}} + (\lambda - 1)\, \underline{e}_1 \otimes \underline{e}_1$$

THUS

$$\underline{\underline{F}} = \nabla \chi = \underline{\underline{A}}$$

THEREFORE $\underline{\underline{F}}$ IS CONSTANT (DOES NOT VARY ACROSS THE BODY). IN THAT CASE IS SAID THAT THE DEFORMATION IS HOMOGENEOUS.

IT IS STRAIGHTFORWARD TO PROVIDE A GEOMETRICAL INTERPRETATION



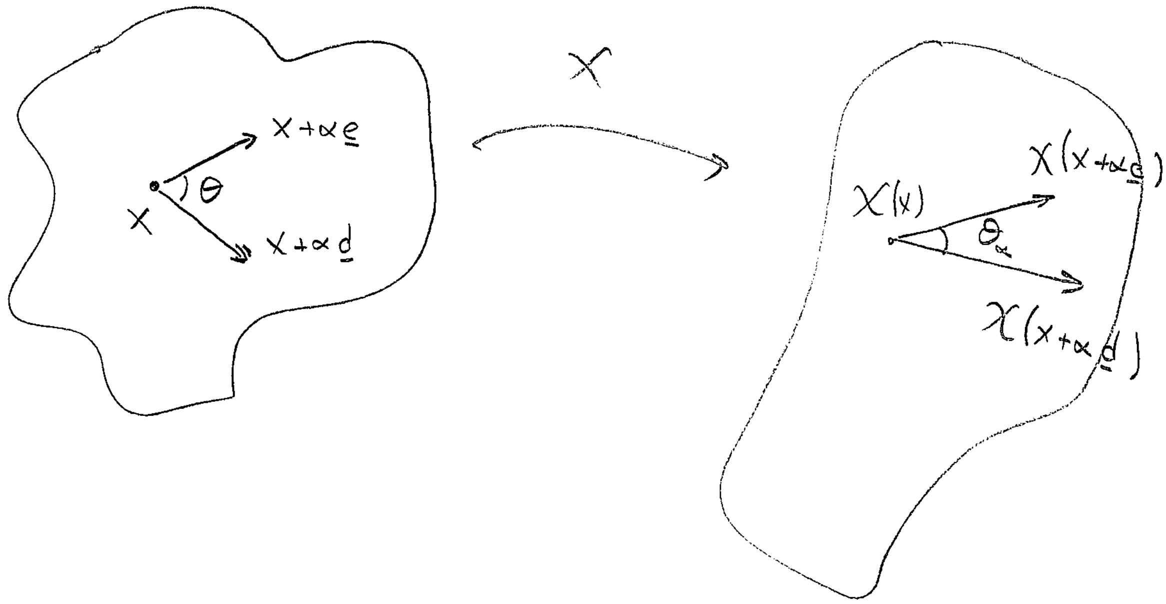
NOW,

$$U^2 = A^T A = A A$$

THEN $\quad A = U$

WHICH SUGGESTS THAT U (OR V) ARE ABLE TO MEASURE STRETCHES IN CERTAIN DIRECTIONS. SO F IS A COMPOSITION OF A ROTATION AND A STRETCH.

WE CAN STILL OBSERVE ANOTHER IMPORTANT COMPONENT OF THE DEFORMATION SCENARIO: ANGULAR DISTORTIONS (NOTE THAT SO FAR WE'VE BEEN TALKING OF STRETCHING SIZE ELEMENTS). IF YOU REMEMBER OF THE EXAMPLE OF DEFORMING THE MEMBRANE, A TYPICAL RECTANGULAR OF THE GRID WOULD HAVE ITS LENGHT'S SIDE MODIFIED BUT ALSO THE ANGLE BETWEN TWO OF THEM COULD CHANGE AFTER DEFORMATION. LET US INVESTIGATE THIS SITUATION INTO MORE DETAIL



$$\cos\theta = \frac{\underline{e} \cdot \underline{d}}{|\underline{e}|\ |\underline{d}|} \qquad (\text{IF } |\underline{e}| = |\underline{d}| = 1) \rightarrow \cos\theta = \underline{e} \cdot \underline{d}$$

AND TAKING $\alpha \rightarrow 0$

$$\frac{\| \chi(X + \alpha \underline{e}) - \chi(X) \|}{|\alpha|} \rightarrow \| U(X)\, \underline{e} \| \qquad (\text{"STRETCH"})$$

AND

$$\cos\theta_\alpha \rightarrow \frac{U\underline{d} \cdot U\underline{e}}{\|U\underline{d}\| \|U\underline{e}\|} \qquad (\text{"ANGULAR DISTORTION"})$$

THE POLAR DECOMPOSITION WAS APPLIED REQUIRING THAT $\det F > 0$. SO, LET $\mathbb{P}$ ANY PART OF $\mathbb{B}$ ITS VOLUME AFTER DEFORMATION IS GIVEN BY

$$\mathrm{Vol}(\chi(\mathbb{P})) = \int_{\chi(\mathbb{P})} dv = \underbrace{\int_{\mathbb{P}} \det F \, dv}_{\text{CHANGE OF VARIABLES}}$$

BY USING, ONCE AGAIN, THE LOCALIZATION THEOREM

$$\det F(x_0) = \lim_{\delta \downarrow 0} \frac{\mathrm{Vol}(\chi(\mathcal{R}_\delta))}{\mathrm{Vol}(\mathcal{R}_\delta)}$$

AND THUS $\det F$ GIVES THE VOLUME AFTER DEFORMATION PER UNIT ORIGINAL VOLUME.

SO $\det F$ <u>MUST</u> BE GREATER THAN 0, OTHERWISE THE BODY WOULD PENETRATE ITSELF WHICH IS PHYSICALLY NOT ACETABLE.

<u>REMARK</u>: WE SAY THAT f IS ISOCHORIC (VOLUME PRESERVING) IF GIVEN ANY PART $\mathbb{P}$

$$\mathrm{vol}(\chi(\mathbb{P})) = \mathrm{vol}(\mathbb{P})$$

WHICH HAS AN IMMEDIATE CONSEQUENCE

A DEF. IS ISOCHORIC $\iff$ $\det F = 1 \quad \forall X$

THE DISPLACEMENT FIELD $\underline{u}$ PLAYS AN IMPORTANT ROLE IN THE DESCRIPTION OF THE MOTION. JUST REMEMBER THAT

$$\chi(x) = X + \underline{u}$$

FROM THAT

$$F = D\chi = I + \nabla \underline{u}$$

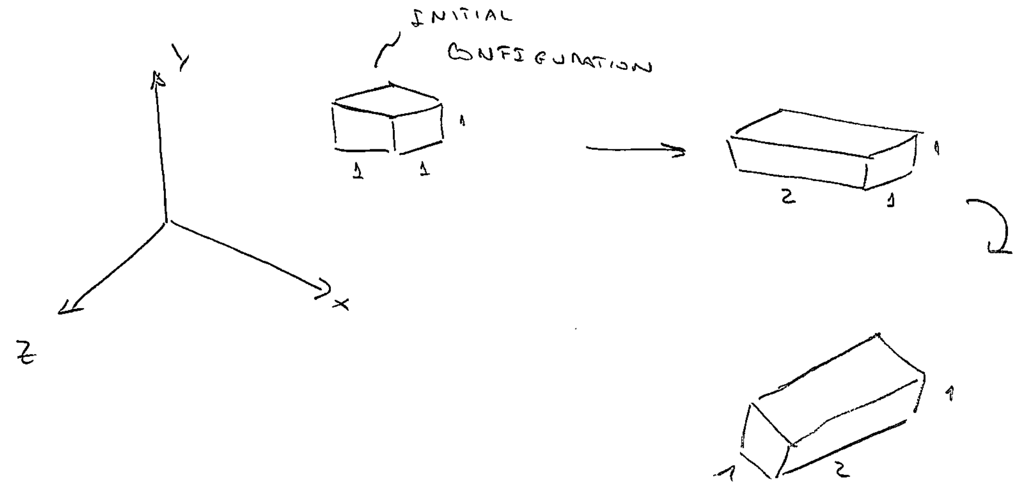
AND

$$E = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T + \nabla \underline{u}^T \nabla \underline{u}\right)$$

<u>REMARK</u>: INDEED $\underline{u}$ OFTEN REPLACES χ IN THE "DEFORMATION" PROBLEM.

Two Simple Examples in "Large Deformations"

(1) Simple Extension (stretch along X direction followed by a 90° rotation about Y axis)



Deformation Description

Initial (reference) configuration:

$$0 \leq X \leq 1 \; ; \; 0 \leq Y \leq 1 \text{ and } 0 \leq Z \leq 1$$

Which is a cube with one vertex coinciding with the origin

(2.13)

LET US FIRST ONLY ANALYZE THE PURE STRECH (LET US ALSO CONSIDER THAT ONE FACE REMAINS ON THE SAME POSITION)

SO:

COORDINATES IN THE DEFORMED CONFIGURATION

$$x = 2X$$
$$y = Y$$
$$z = Z$$

THUS $F = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow F^T F = \begin{bmatrix} 4 & 0 & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

AND $U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

SO $E = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

AND $\det F = 2$ (THE VOLUME OF THE DEFORMED CUBE IS TWICE THE ORIGINAL ONE)

MOREOVER,

$$\underline{u}(X,Y,Z) = X\,\underline{e}_1 \quad \text{AND} \quad D\underline{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

NOTE THAT WE HAVE AN HOMOGENEOUS DEFORMATION, WHICH DOES NOT MEAN THAT EACH PARTICLE HAS MOVED IN THE SAME WAY

LET US NOW MOVE FORWARD AND TAKE INTO CONSIDERATION THE ROTATION TOO (KEEPING FIXED THE EDGE COINCIDING WITH THE Y-AXIS)

$$\left.\begin{aligned} x &= -Z \\ y &= Y \\ z &= 2X \end{aligned}\right\}$$

(IT IS JUST GEOMETRY - MAKE A SKETCH CORRESPONDING TO THIS SITUATION)

THEN,

$$F = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

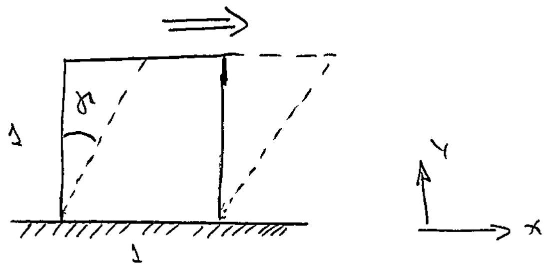
NOTE THAT $\det F = 2$ (AS IT SHOULD BE!) AND

AS $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (DO YOU UNDERSTAND WHY?)

$$F = RU \longrightarrow U = R^T F = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

AND $E = \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(2) SIMPLE SHEAR



INITIAL CONFIGURATION – CUBE ALIGNED WITH X, Y, Z AXIS (CONTINUOUS LINE DRAWING)

DEFORMED CONFIGURATION – INCLINED PARALLELOGRAM (DASHED LINES)

$$x = X + Y \tan\gamma$$

$$y = Y$$

$$z = Z$$

THUS:

$$F = \begin{bmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.17

$$F^T F = \begin{bmatrix} 1 & \tan\gamma & 0 \\ \tan\gamma & 1+\tan^2\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det F = 1$ (SO IT IS AN ISOCHORIC DEFORMATION)

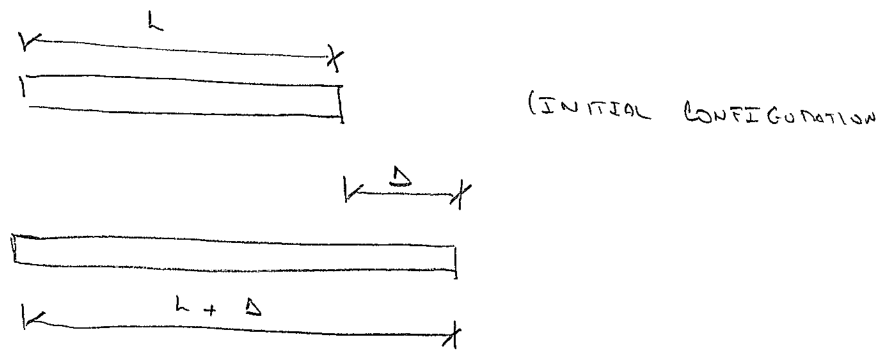
AND

$\underline{u}(X, Y, Z) = Y \tan\gamma \, \underline{e_1}$

WHAT ABOUT U AND R?

2.2) SMALL DEFORMATIONS

PROBABLY THE SIMPLEST MANIFESTATION OF DEFORMATION (OR STRAIN) IS THE ELONGATION OF A BAR AS DEPICTED IN THE FIGURE BELLOW



* ENGINEERING CONCEPTS OF STRAIN

$$\lambda = \frac{L+\Delta}{L}$$

$$\varepsilon_{(1)} = \frac{\Delta}{L} = \lambda - 1 \quad \text{———} \quad U - I$$

EXTENSION FOR 3D

$$\varepsilon_{(2)} = \frac{(L+\Delta)^2 - L^2}{2L^2} = \frac{1}{2}\left(\lambda^2 - 1\right) \quad \text{———} \quad E$$

IMPORTANT : $\varepsilon_{(1)}$ IS A LINEAR MEASURE OF STRAIN $\left(\text{OR } \lambda = \frac{L+\Delta}{L}\right)$

NOTE ALSO THAT $\lambda^2 \sim 1 + 2(\lambda - 1)$
(λ AROUND 1)

THEN $\varepsilon_{(2)} \sim \varepsilon_{(1)}$

SO, WHEN THE GEOMETRY OF THE FINAL (DEFORMED) CONFIGURATION DOES NOT "CHANGE MUCH", $\varepsilon_{(1)}$ CAN REPLACE $\varepsilon_{(2)}$, WHICH LEADS TO A "LINEARIZED" DESCRIPTION OF THE DEFORMATION.

NOTE THAT $\underline{u}(x) = (\lambda - 1) X \longrightarrow \frac{d\underline{u}}{dx} = \lambda$

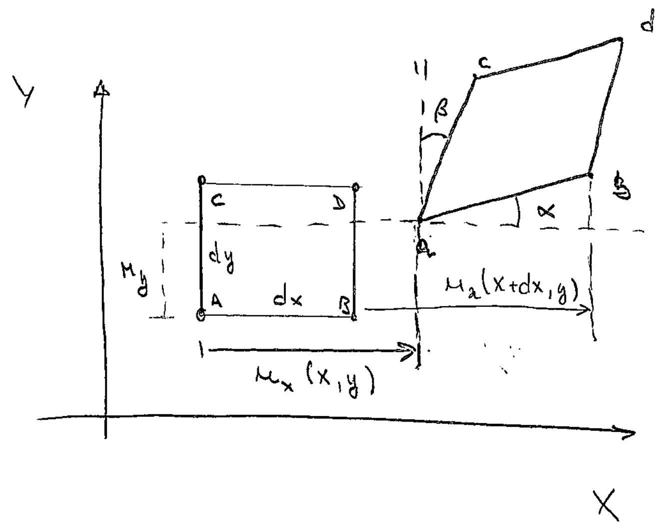
WHICH SUGESTS FOR THE GENERAL 3-D CASE : SMALL STRAINS $\sim$ $|\nabla \underline{u}| \ll 1$, THEREFORE

$$E = \frac{1}{2} \left(\nabla \underline{u} + \nabla^T \underline{u} + \nabla^T \underline{u} \, \nabla u \right)$$

$$\hookrightarrow \quad \varepsilon = \frac{1}{2} \left(\nabla \underline{u} + \nabla^T \underline{u} \right) \qquad \left(\text{AS } \nabla^T u \, \nabla u \searrow 0 \right)$$

THE COMPONENTS ε_{ij} ARE REFERRED TO AS SMALL OR INFINITESIMAL STRAINS

THE ABOVE MEASURE OF STRAINS CAN BE BETTER UNDERSTOOD IF INTERPRETED IN A GEOMETRICAL SETTING



CLEARLY, THE DEFORMATION OF THE INFINITESIMAL RECTANGLE (dx, dy) IS GIVEN BY A COMPOSITION OF EXTENSIONAL STRAINS (STRETCHES) AND ANGLES DISTORTION, OFTEN REFERRED TO AS SHEAR STRAINS.

$$\varepsilon_x = \frac{ab - AB}{AB}$$

WHICH, DIRECTLY FROM THE GEOMETRY, (NOTE ALSO $u_x(x+dx,y) \simeq u_x(x) + \frac{du_x}{dx}dx$)

$$ab = \sqrt{\left(dx + \frac{\partial u_x}{\partial x}dx\right)^2 + \left(\frac{\partial u_y}{\partial x}dx\right)^2} =$$

$$\sqrt{1 + 2\frac{\partial u_x}{\partial x} + \left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2}\, dx \simeq \left(1 + \frac{\partial u_x}{\partial x}\right) dx$$

(IF $\frac{\partial u_x}{\partial x}$ AND $\frac{\partial u_y}{\partial x}$ ARE SMALL)

THEN $\varepsilon_x = \dfrac{\partial u_x}{\partial x}$

(AN ANALOGUE DEVELOPMENT LEADS TO $\varepsilon_y = \dfrac{\partial u_y}{\partial y}$)

P.S: TO BE CONSISTENT WITH THE ADOPTED NOTATION WE SHOULD USE ε_{xx} INSTEAD ε_x ...

THE SECOND TYPE OF DEFORMATION IS RELATED TO ANGLES α AND β AND CAN BE EXPRESSED AS

$$\gamma_{xy} = \frac{\pi}{2} - \angle cab = \alpha + \beta$$

FOR SMALL DEFORMATIONS: $\alpha \sim \tan\alpha$ AND $\beta \sim \tan\beta$, THUS

$$\gamma_{xy} = \frac{\frac{\partial u_y}{\partial x}\, dx}{dx + \frac{\partial u_x}{\partial x}\, dx} + \frac{\frac{\partial u_x}{\partial y}\, dy}{dy + \frac{\partial u_y}{\partial y}\, dy} =$$

$$= \frac{\frac{\partial u_y}{\partial x}\, dx}{dx\,(1 + \varepsilon_x)} + \frac{\frac{\partial u_x}{\partial y}\, dy}{dy\,(1 + \varepsilon_y)} \sim \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

IT IS STRAIGHTFORWARD TO PROVE

$$\gamma_{xy} = \gamma_{yx}$$

AND $$\gamma_{xy} = 2\,\varepsilon_{xy}$$

P.S.: (1) IN PARTICULAR APPLICATIONS IT IS CONVENIENT TO DECOMPOSE THE STRAIN TENSOR INTO TWO PARTS CALLED SPHERICAL AND DEVIATORIC STRAIN TENSORS. THE SPHERICAL PART IS DEFINED AS

$$\tilde{\varepsilon} = \frac{1}{3}\,tr(\varepsilon)\,I$$

WHILE THE DEVIATORIC STRAIN IS GIVEN BY

$$\hat{\varepsilon} = \varepsilon - \tilde{\varepsilon}$$

THE SPHERICAL STRAIN IS ASSOCIATE TO VOLUMETRIC DEFORMATION AND IS AN ISOTROPIC TENSOR (THE COMPONENTS REMAIN THE SAME UNDER ANY TRANSFORMATION OF COORDINATES). THE DEVIATORIC STRAIN TENSOR ACCOUNTS FOR CHANGES IN SHAPE OF "MATERIAL ELEMENTS".

INDEED, WE KNOW THAT $\frac{V}{V_0} = \det F = \begin{vmatrix} 1+\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & 1+\frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial z} & \frac{\partial u_z}{\partial y} & 1+\frac{\partial u_z}{\partial z} \end{vmatrix}$

IF $\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1$

$$\frac{V}{V_0} = 1 + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} + O\left(\frac{\partial u_i}{\partial x_j}\right) \approx 1 + tr(\varepsilon)$$

THUS $\frac{V-V_0}{V_0} = tr(\varepsilon) = tr(\tilde{\varepsilon})$

(ii) LIKE VECTORS, TENSORS ARE REPRESENTED BY DIFFERENT COMPONENTS ON DIFFERENT COORDINATES SYSTEMS. INDEED, LET S BE A TENSOR AND $[S]_1$ ITS REPRESENTATION ON THE COORDINATE FRAME-1, THEN $[S]_2 = Q^T [S]_1 Q$ IS THE MATRIX REPRESENTING S ON THE COORDINATE FRAME-2. Q BEING THE MATRIX (ROTATION) CORRESPONDING TO THE CHANGE OF COORDINATES. SO IT IS REASONABLE TO EXPECT THAT IN A SPECIFIC COORDINATE SYSTEM WE HAVE MAXIMUM VALUES.

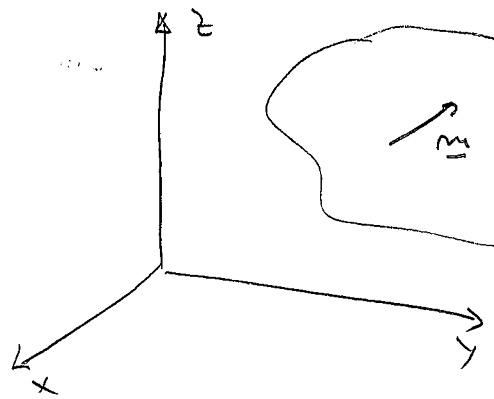
IN THE CASE OF THE STRAIN TENSOR (SYMMETRIC TENSOR) THEY ARE GIVEN BY

$$\varepsilon \, \underline{v}_i = \lambda_i \underline{v}_i \qquad i = 1, 2, 3$$

THE EINGEN VALUES OF ε.

MOREOVER, IN THE COORDINATE SYSTEM STEMMING FROM THE EIGENVECTORS DIRECTIONS, $[\varepsilon]$ IS A DIAGONAL MATRIX.

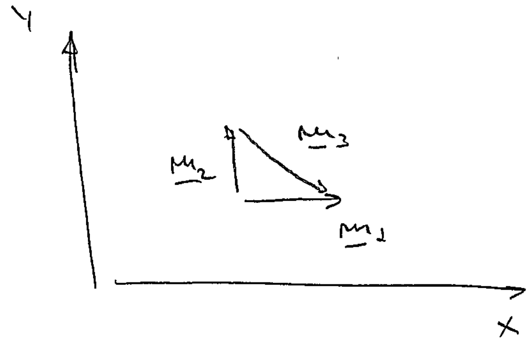
LET US PICK ANY DIRECTION (NOT COINCIDENT WITH ONE OF THE INITIAL COORDINATE FRAME)



THEN,

$$\underline{m} \cdot E\, \underline{m} = \frac{1}{2}\left(\lambda^2 - 1\right) \sim \lambda - 1 = \underline{m} \cdot \varepsilon\, \underline{m}$$

P.S: STRAIN GAGES ALIGNED ALONG DIFFERENT DIRECTIONS AT A SOLID SURFACE CAN BE USED TO MEASURE STRAIN IN THE PLANE SURFACE, SUCH THAT



$\underline{m}_i$ – $(\text{STRAIN GAGE})_i$

THUS $\underline{m}_1 \cdot \mathcal{E}\, \underline{m}_1 = \mathcal{E}_{11} = \mathcal{E}_1$; $\underline{m}_2 \cdot \mathcal{E}\, \underline{m}_2 = \mathcal{E}_{22} = \mathcal{E}_2$

AND THE SHEAR STRAIN $\mathcal{E}_{12}$ IS TO BE DETERMINED WITH THE HELP OF THE READING IN THE $\underline{m}_3$ DIRECTION.

FOR THE PLANE CASE

$$[\mathcal{E}] = \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{12} & \mathcal{E}_{22} \end{bmatrix}$$

(MATRIX REPRESENTATION IN THE FIXED COORDINATE FRAME)

KNOWN FROM DIRECT READINGS

LET Q THE ROTATION MATRIX LINKING THE INITIAL COORDINATE FRAME AND THE ONE CONTAING $\underline{m}_3$ AS A COORDINATE DIRECTION. (THAT WOULD MAKE AN EXCELENT EXERCISE BUILDING Q)

THAN

$$[\varepsilon]' = Q^T [\varepsilon] Q$$

↳ THE MATRIX REPRESENTATION IN THE "NEW" COORDINATE FRAME. THUS

$$\varepsilon_3 = \underline{m}_3 \cdot [\varepsilon]' \underline{m}_3 = f(\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \theta)$$

THE READING FROM STRAIN-GAGE 3

KNOWN ROTATION ANGLE LINKING THE BOTH COORDINATE SYSTEMS

THUS, ONE HAS TO SOLVE FOR ε_{12} THE ALGEBRAIC EQUATION

$$\varepsilon_3 - f(\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \theta) = 0$$

(iii) COMPATIBILITY EQUATIONS:

$$\underset{\sim}{u} \longrightarrow \varepsilon = \frac{1}{2}\left(\nabla \underset{\sim}{u} + \nabla \underset{\sim}{u}^T\right)$$

3 DISPLACEMENTS COMPONENTS

6 STRAIN COMPONENTS

THE QUESTION THAT ARISES IN THE FOLLOWING IS: HOW TO OBTAIN (IF POSSIBLE!) THE DISPLACEMENTS FROM STRAINS? (PLEASE NOTE THE IMPORTANCE OF THIS QUESTION BY JUSTING COME A LITTLE WAY BACK IN THESE NOTES: YOU CAN MEASURE DEFORMATIONS!)

THE MAIN ISSUE: YOU NEED TO HAVE MORE 3 EQUATIONS CONSTRAINING THE STRAIN COMPONENTS, THE SO CALLED COMPATIBILITY CONDITIONS:

FOR EXAMPLE:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2\,\frac{\partial^2 \varepsilon_{xy}}{\partial x\,\partial y}$$

(JUST PLUG IN THE ABOVE EQUATION THE VERY DEFINITIONS OF ε_{ij} AND CHECK)

Similarly

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = 2 \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}$$

The only remaining aspect involving the computation of the displacement field from the strain tensor comes from the integration constants (note that $\varepsilon_{ij} = f\left(\frac{\partial u_i}{\partial x_j}\right)$). They are sometimes related to rigid body motions superposed to deformations. You will have the opportunity of dealing with this situation in the next homework.