1.3) TENSOR ANALYSIS: DIFFERENTIATION AND INTEGRAL THEOREMS

WE INTRODUCE NOW A NOTION OF DIFFERENTIATION SUFFICIENTLY GENERAL TO INCLUDE SCALAR, VECTOR OR TENSOR FUNCTIONS WHOSE ARGUMENTS ARE SCALARS, VECTORS OR TENSORS. ALL WE NEED FOR STARTING IS TO BE ABLE TO DEFINE A NEIGHBORHOOD AROUND A POINT ON THE FUNCTION'S DOMAIN.

LET g BE A FUNCTION DEFINED SUCH THAT

$$g : D \to W$$

WHERE D AND W ARE SETS ENDOWED WITH A WAY OF MEASURING DISTANCES, THEN g IS DIFFERENTIABLE AT x (NOTE THA x CAN BE EITHER A SCALAR, VECTOR, POINT OR TENSOR) IF THERE EXISTS A LINEAR TRANSFORMATION

$$Dg(x) : D \to W$$

SUCH THAT

$$\boxed{g(x+u) = g(x) + Dg(x)[u] + o(u)}$$

AS $u \to 0$

WHERE $o(u)$ IS SAID TO APPROACH ZERO FASTER THAN u IF

$$\lim_{\substack{u \to 0 \\ u \neq 0}} \frac{\|f(u)\|}{\|u\|} = 0$$

REMARKS (1) AS THE DEFINITION IS GENERAL, THE USUAL NOTATION FOR REMARKING DIFFERENCES AMONG VECTORS, TENSORS, OR SCALARS IS NOT NECESSARY

(2) Dg MIGHT NOT EXIST (NON DIFFERENTIABLE FUNCTION AT x), BUT IF IT EXISTS IT IS UNIQUE

(3) INDEED, THERE ARE IMPORTANT SITUATIONS IN WHICH THE DOMAIN OF g AND Dg DOES NOT COINCIDE.

(4) THE DEFINITION OF Dg CAN BE REPHRASED IN ORDER TO RETRIEVE THE USUAL CONCEPT OF DIRECTIONAL DERIVATIVE:

$$Dg(x)[u] = \lim_{\substack{\alpha \to 0 \\ \alpha \in \mathbb{R}}} \frac{1}{\alpha}\left[g(x+\alpha u) - g(x)\right] = \frac{d}{d\alpha} g(x+\alpha u)\Big|_{\alpha=0}$$

EXAMPLES:

(1) CONSIDER THE FUNCTION

$$\varphi : \mathbb{V} \to \mathbb{R}$$

"VECTOR SPACE"

DEFINED BY: $\varphi(\underline{v}) = \underline{v} \cdot \underline{v}$ (THIS COULD REPRESENT KINETIC ENERGY)

THEN

$$\varphi(\underline{v}+\underline{u}) = (\underline{v}+\underline{u}) \cdot (\underline{v}+\underline{u}) = \underline{v}\cdot\underline{v} + 2\underline{v}\cdot\underline{u} + \underline{u}\cdot\underline{u} =$$

$$= \varphi(\underline{v}) + 2\underline{v}\cdot\underline{u} + \overbrace{o(\underline{u})}^{\underline{u}\cdot\underline{u}}$$

SO $D\varphi(\underline{v})[\underline{u}] = 2\underline{v}\cdot\underline{u}$

REMARKS: (i) $o(\underline{u}) = \underline{u}\cdot\underline{u}$ IS SUCH THAT

$$\lim_{\|\underline{u}\|\downarrow 0} \frac{\underline{u}\cdot\underline{u}}{\|\underline{u}\|} = \lim_{\|\underline{u}\|\downarrow 0} \frac{\|\underline{u}\|^2}{\|\underline{u}\|} = \lim_{\|\underline{u}\|\downarrow 0} \|\underline{u}\| = 0$$

(ii) JUST TO MAKE SURE YOU HAVE REALLY UNDERSTOOD WHAT WE DID RIGHT BELOW:

"WE ARRIVED TO THE DERIVATIVE OF A SCALAR FUNCTION WITH RESPECT TO ITS VECTOR ARGUMENT.

(2) $G(A) = A^2$ (WHERE A IS A TENSOR)

THUS WE HAVE:

$$G(A+U) = A^2 + AU + UA + U^2$$

IN WHICH WE RECOGNIZE

$$G(A+U) = G(A) + AU + UA + o(U^2)$$

THEN $$DG(A)[U] = AU + UA$$

(3) $\varphi(A) = \det(A)$ (WHERE A IS A TENSOR)

ONE CAN PROVE THAT

$$\det(I+S) = 1 + \operatorname{tr} S + o(S)$$

AS $S \to 0$

So, ASSUMING A IS INVERTIBLE ($\det A \neq 0$)

$$\det(A+U) = \det[(I+UA^{-1})A] = (\det A)(\det(I+UA^{-1})) =$$

$$= \det A\, [1 + \operatorname{tr}(I+UA^{-1}) + o(U)] =$$

$$= \det A + (\det A)\operatorname{tr}(UA^{-1}) + o(U)]$$

$$D\varphi(A)[U] = (\det A)\operatorname{tr}(UA^{-1})$$ AS $U \to 0$

GRADIENT, DIVERGENCE AN CURL

(LET'S GO BACK TO YOURS CALCULUS COURSES)

- WE NOW CONSIDER ONLY FUNCTIONS DEFINED OVER AN EUCLIDIAN $\mathcal{E}$ 3-D SPACE (THEREFORE THEY ARE "FUNCTIONS OF POINTS x"). THESE FUNCTIONS ARE CALLED SCALAR, VECTOR OR TENSORS ACCORDING TO THE KIND OF VALUE THEY LEAD TO.

LET φ A SCALAR FUNCTION (REMEMBER $\varphi: \mathcal{E} \to \mathbb{R}$).

THEN

THIS IS A VECTOR DEFINED AT EACH x

$$D\varphi(x)[\underline{u}] = \underline{\nabla\varphi(x)} \cdot \underline{u}$$

GRADIENT OF φ AT POINT x

(THE ABOVE EQUALITY IS A DIRECT RESULT FROM THE RIEZ REPRESENTATION THEOREM)

THUS

$$\varphi(x+u) = \varphi(x) + \nabla\varphi \cdot u + o(u)$$

REMARK: AT THAT POINT IT IS USEFULL TO HAVE IN MIND THE TAYLOR'S SERIE EXPANSION.

\# SIMILARLY, IF $\underline{v}$ IS A SMOOTH VECTOR FUNCTION, THEN $D\underline{v}(x)$ IS A LINEAR TRANSFORMATION. (A TENSOR). IN THAT CASE

$$D\underline{v}(x)[\underline{u}] = D\underline{v}(x)\,\underline{u}$$

THE TENSOR $\nabla\underline{v}(x)$ IS THE GRADIENT OF $\underline{v}$ AT x.

DEF.: DIVERGENCE

$$\operatorname{div} \underline{v} = \operatorname{tr}(\nabla \underline{v})$$

REMARK: THE ABOVE DEFINITION CAN BE EXTENDED TO TENSOR FIELDS (SYNONIMOUS OF FUNCTIONS) SUCH THAT

$$(\underline{\operatorname{div} S}) \cdot \underline{a} = \operatorname{div}(S^T \underline{a})$$

WHERE S IS THE TENSOR FIELD

HEREAFTER, A LIST OF IMPORTANT RESULTS INVOLVING DERIVATIVES IS PRESENTED. LET φ, $\underline{v}$, $\underline{w}$ AN S BE SMOOTH FIELDS WITH φ SCALAR VALUED, $\underline{v}$ AND $\underline{w}$ VECTOR VALUED AND S TENSOR VALUED. THEN:

$$\nabla(\varphi \underline{v}) = \varphi \nabla \underline{v} + \underline{v} \otimes \underline{\nabla \varphi}$$

$$\operatorname{div}(\varphi \underline{v}) = \varphi \operatorname{div} \underline{v} + \underline{v} \cdot \underline{\nabla \varphi}$$

$$\nabla(\underline{v} \cdot \underline{w}) = (\nabla \underline{w})^T \underline{v} + (\nabla \underline{v})^T \underline{w}$$

$$\operatorname{div}(\underline{v} \otimes \underline{w}) = \underline{v} \operatorname{div} \underline{w} + (\nabla \underline{v})\, \underline{w}$$

$$\operatorname{div}(S^T \underline{v}) = S \cdot \nabla \underline{v} + \underline{v} \cdot \operatorname{div} S$$

$$\operatorname{div}(\varphi S) = \varphi \operatorname{div} S + S \nabla \varphi$$

THE ABOVE RELATIONS RELY UPON THE FOLLOWING RESULT (PRODUCT RULE) STATED BELLOW WITHOUT PROOF

LET f AND g BE DIFFERENTIABLE AT x. THEN THEIR PRODUCT $h = \pi(f, g)$ IS DIFFERENTIABLE AT x (WHERE π STANDS FOR ONE OF THE MANY PRODUCT FORMS WE HAVE BEEN DEALING WITH)

$$\boxed{D h(x)[u] = \pi\big(f(x), Dg(x)[u]\big) + \pi\big(Df(x)[u], g(x)\big)}$$

DEF.: THE CURL OF $\underline{v}$, DENOTED $\operatorname{curl} \underline{v}$ OR $\operatorname{rot} \underline{v}$, IS THE UNIQUE VECTOR FIELD WITH THE PROPERTY

$$(\nabla \underline{v} - \nabla \underline{v}^T)\, \underline{a} = (\operatorname{curl} \underline{v}) \times \underline{a}$$

$$\forall\, \underline{a}$$

INDEED, $\operatorname{curl} \underline{v}(x)$ IS THE AXIAL VECTOR CORRESPONDING TO THE SKEW TENSOR $\nabla \underline{v} - \nabla \underline{v}^T$ (PLEASE NOTE THAT

$$(\nabla \underline{v} - \nabla \underline{v}^T)^T = \nabla \underline{v}^T - \nabla \underline{v} = -(\nabla \underline{v} - \nabla \underline{v})\,)$$

LET US PHRASE THE ABOVE "DERIVATIVES" IN COORDINATES (THE COORDINATE FRAME IS GIVEN BY x_i $i = 1, .. 3$)

$$(\nabla \varphi)_i = \frac{\partial \varphi}{\partial x_i} \qquad (\nabla \underline{v})_{ij} = \frac{\partial v_i}{\partial x_j}$$

$$\operatorname{div} \underline{v} = \sum_i \frac{\partial v_i}{\partial x_i} \qquad (\operatorname{div} S)_i = \sum_j \frac{\partial S_{ij}}{\partial x_j}$$

$$(\operatorname{curl} \underline{v})_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \;; \quad (\operatorname{curl} \underline{v})_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}$$

$$(\operatorname{curl} \underline{v})_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$$

- CHAIN RULE: LET f AND g BE DIFFERENTIABLE AT x AND $y = g(x)$ THEN THE COMPOSITION

$$h = f \circ g = f(g(x))$$

IS DIFFERENTIABLE AT x AND

$$Dh(x) = Df(y) \circ Dg(x)$$

THE ABOVE GENERAL RESULT YIELDS, IN THE CASE OF f AND g SCALAR FUNCTIONS, TO:

$$\nabla h(x) = \underline{\nabla f}(y)\ \underline{\nabla g}(x)$$

REMARK: (POTENTIAL THEOREM) LET $\underline{v}$ BE A SMOOTH VECTOR FIELD AND CURL $\underline{v} = 0$. THEN THERE IS A CLASS C^2 SCALAR FIELD φ SUCH THAT

$$\underline{v} = \nabla \varphi$$

(C^2 MEANS THAT THE FUNCTION HAS TWO DERIVATIVES)

EXAMPLES:

(i) LET θ BE THE TEMPERATURE FIELD DEFINED OVER THE BODY Ω SUCH THAT $\theta = \operatorname{Sen}(x_1 t)$. COMPUTE SPACE AND TIME "DERIVATIVES" OF θ

\# TIME $\quad \dfrac{\partial \theta}{\partial t} = \cos(x_1 t)\, x_1$

\# SPACE $\quad \underline{\nabla \theta} = \left(\cos(x_1 t), 0, 0 \right) t$

(ii) OBTAIN, FROM THE DEFINITION, THE EXPRESSION FOR THE COMPONENTS OF THE GRADIENT OF A SCALAR FUNCTION

SOL. $\quad \varphi(x + \underline{u}) = \varphi(x) + \underline{\nabla \varphi} \cdot \underline{u} + O(u) \quad \forall \underline{u}$

IF $\quad \underline{u} = \Delta x_2\, \underline{e_2}$

$$\varphi(x_1, x_2 + \Delta x_2, x_3) = \varphi(x_1, x_2, x_3) + \Delta x_2 \left(\underline{\nabla \varphi} \right)_2 + O(\Delta x_2)$$

$$\longrightarrow \frac{\partial \varphi}{\partial x_2}$$

REMARK: IT IS STRAIGHTFORWARD TO ASSESS HOW A SCALAR FUNCTION VARIES IN A CERTAIN DIRECTION $\underline{m}$:

$$\frac{\partial \varphi}{\partial m} = \nabla \underline{\varphi} \cdot \underline{m}$$

$\hookrightarrow$ "DIRECTIONAL DERIVATIVE"

(iii) COMPUTE THE GRADIENT OF $\phi(x) = Ax \cdot x$ (A IS A CONSTANT TENSOR)

SOL:

$$\phi(x+\underline{u}) = A(x+\underline{u}) \cdot (x+\underline{u}) = Ax \cdot x + A\underline{u} \cdot x + Ax \cdot \underline{u} + A\underline{u} \cdot \underline{u}$$

$$= \phi(x) + (A + A^T)x \cdot \underline{u} + O(\underline{u})$$

THEREFORE $\nabla \phi = (A + A^T)x$

THE SAME RESULT COULD BE OBTAINED THROUGH THE USE OF COMPONENTS

$$\phi = A_{ij} x_j x_i$$

CALCULATING THE PARTIAL DERIVATIVE OF ϕ WITH RESPECT TO x_k YIELDS

$$\phi_{,k} = A_{ij}(x_i x_k)_{,k} = A_{ij}(x_{i,k} x_j + x_i x_{j,k}) = A_{ij}(\delta_{ik} x_j + x_i \delta_{jk})$$

$\Downarrow$

NOTATION

$$\frac{\partial \phi}{\partial x_k}$$

$\delta_{k\ell} \rightarrow$ KRONECKER DELTA

$$\delta_{kl} \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

THUS $\phi_{,k} = A_{kj} x_j + A_{ik} x_i = \underbrace{(A_{kj} + A_{jk})} x_j$

$$\longrightarrow \nabla \Phi = (A + A^T) x$$

(iv) LET $\underline{u}(x)$, $E(x)$ AND $S(x)$ BE, RESPECTIVELY, A VECTOR AND TWO TENSOR FIELDS. THESE FIELDS ARE RELATED BY

$$E = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T \right), \quad S = 2\mu E + \lambda \, tr(E) \, 1$$

WHERE λ AND μ ARE CONSTANTS. SUPPOSE THAT

$$\underline{u}(x) = \frac{b}{r^3} x \qquad \text{WHERE} \quad r = \|x\| , |x| \neq 0$$

WITH b A CONSTANT

$$\left(u_1 = \frac{b x_1}{r^3} ; \quad u_2 = \frac{b}{r^3} x_2 ; \quad u_3 = \frac{b}{r^3} x_3 \right)$$

$$u_i = \frac{b}{r^3} x_i$$

COMPUTE $E(x)$ AND $S(x)$. AFTER VERIFY THAT THE FIELD $S(x)$ SATISFIES THE EQUATION

$$\operatorname{div} S = 0 \qquad |x| \neq 0$$

BEFORE COMPUTING $\frac{\partial u_i}{\partial x_j}$, OBSERVE BY DIFFERENTIATING $r^2 = \|x\|^2 = x_i x_i$ THAT

$$2 r_j \frac{\partial r}{\partial x_j} = 2 \frac{\partial x_i}{\partial x_j} x_i = 2 \delta_{ij} x_i = 2 x_j$$

$$\longrightarrow \frac{\partial r}{\partial x_j} = \frac{x_j}{r}$$

THEREFORE

$$\frac{\partial u_i}{\partial x_j} = b \left\{ \frac{\delta_{ij}}{r^3} - 3 \frac{x_i x_j}{r^5} \right\}$$

THEN

$$E = \frac{b}{r^3} \begin{bmatrix} 1 - \frac{3 x_1^2}{r^2} & -\frac{3 x_1 x_2}{r^2} & -3 \frac{x_1 x_3}{r^2} \\ & 1 - \frac{3 x_2^2}{r^2} & 1 - \frac{3 x_2 x_3}{r^2} \\ \text{SIM} & & \\ & & 1 - \frac{3 x_3^2}{r^2} \end{bmatrix}$$

Now

$$tr(E) = \frac{b}{r^3}\left[3 - 3\right] = 0$$

$$\left(\text{NOTE THAT } \frac{3x_1^2}{r^2} + \frac{3x_2^2}{r^2} + \frac{3x_3^2}{r^2} = 3\frac{\cancel{r^2}}{\cancel{r^2}}\right)$$

AND THEN

$$S = 2\mu E = \frac{2\mu b}{r^3}\begin{bmatrix} 1 - 3\frac{x_1^2}{r^2} & \cdot & \cdot \\ \vdots & \cdot & \cdots \\ \vdots & - & \vdots \end{bmatrix}$$

FINALLY, $\operatorname{div} S \Longrightarrow \frac{\partial S_{ij}}{\partial x_j} = \frac{\partial S_{11}}{\partial x_1} + \frac{\partial S_{12}}{\partial x_2} + \frac{\partial S_{13}}{\partial x_3} +$

$$\frac{\partial S_{21}}{\partial x_1} + \frac{\partial S_{22}}{\partial x_2} + \frac{\partial S_{23}}{\partial x_3} + \frac{\partial S_{31}}{\partial x_1} + \frac{\partial S_{32}}{\partial x_2} + \frac{\partial S_{33}}{\partial x_3}$$

FOR $i = j \longrightarrow \frac{\partial S_{ii}}{\partial x_i} = 2\mu\left[-\frac{3}{r^4}\frac{x_i}{r} - \frac{6x_i}{r^5} + 15\frac{x_i^3}{r^7}\right]$

FOR $i \neq j \longrightarrow \frac{\partial S_{ij}}{\partial x_j} = 2\mu\left[-\frac{6x_i}{r^5} - \frac{15}{r^7}x_i x_j^2\right]$

$\longrightarrow \operatorname{div} S = 0$

THE DIVERGENCE THEOREM

CONSIDER A REGION Ω OF $\mathcal{E}$ (WHICH VOLUME IS DENOTED AS $\mathrm{vol}(\Omega)$) THEN

LET $\varphi : \Omega \to \mathbb{R}$, $\underline{v} : \Omega \to \underset{\substack{\text{VECTOR}\\ \text{SPACE}}}{\mathcal{V}}$ AND $\overset{\text{TENSOR}}{S} : \Omega \to \underset{\substack{\text{TENSOR}\\ \text{SPACE}}}{\mathrm{Lin}}$

SMOOTH FIELDS, THEN

$$\int_{\partial\Omega} \varphi \, \underline{n} \, dA = \int_{\Omega} \underline{\nabla\varphi} \, dV$$

$$\int_{\partial\Omega} \underline{v} \cdot \underline{n} \, dA = \int_{\Omega} \mathrm{div}\, \underline{v} \, dV$$

$$\int_{\partial\Omega} S \underline{n} \, dA = \int_{\Omega} \mathrm{div}\, S \, dV$$

WHERE $\underline{n}$ IS THE OUTWARD UNIT NORMAL ON THE BORDER SURFACE $\partial\Omega$.

THE ABOVE THEOREM, (STATED WITHOUT PROOF) SHOWS ITS IMPORTANCE WHEN WE DERIVE THE BALANCE LAWS THAT GOVERN THE MECHANICS OF A ELASTIC BODY. AIMING AT THAT DERIVATION WE ALSO NEED THE FOLLOWING RESULT

LOCALIZATION THEOREM: LET $\underline{\Phi}$ BE A CONTINUOUS SCALAR OR VECTOR FIELD ON Ω. THEN GIVEN ANY $x_0 \in \Omega$

$$\underline{\Phi}(x_0) = \lim_{\delta \to 0} \frac{1}{Vol(\Omega_\delta)} \int_{\Omega_\delta} \underline{\Phi} \, dV$$

WHERE Ω_δ $(\delta > 0)$ IS A "BALL" OF RADIUS δ CENTERED AT x_0. THEREFORE, IF $\int_{\Omega} \underline{\Phi} \, dV = 0$ $\forall \, \Omega_\delta \subset \Omega$ THEN

$$\boxed{\underline{\Phi} = 0}$$

REMARK: COMBINING LOCALIZATION AND DIVERGENCE THEOREMS YIELD THE FOLLOWING INTERESTING INTERPRETATION OF THE DIVERGENCE:

$$\operatorname{div} \underline{v}(x) = \lim_{\delta \to 0} \frac{1}{\operatorname{Vol}(R_\delta)} \int \operatorname{div} \underline{v} \, dV =$$

$$= \lim_{\delta \to 0} \frac{1}{\operatorname{vol}(R_\delta)} \int \underline{v} \cdot \underline{n} \, dA$$

- PLEASE NOTE, A SIMILAR RESULT APPLIES TO THE TENSOR FUNCTIONS.

- NOTE THAT $\displaystyle\int_{R_\delta} v \cdot n \, dA = 0 \iff \operatorname{div} \underline{v} = 0$

Example: Let $A(x)$ be a tensor field (note that this is broader version of the example presented in class which considered A as a constant ...) which satisfies $\operatorname{div} A = 0$ at each point of the domain $\mathbb{D}$. Assume also that

$$\int_{\partial P} x \wedge A\underline{m} \, d\Gamma = \underline{0} \quad \forall \text{ subregions } P \subset \mathbb{D}$$

Vector product

Show that A must be a symmetric tensor (e.g. $A_{ij} = A_{ji}$)

Dem: In terms of components

$$\int_{\partial P} e_{ijk} \, x_j \, (A\underline{m})_k \, d\Gamma = \int_{\partial P} e_{ijk} \, x_j \, A_{kp} m_p \, d\Gamma = 0$$

By noting that for fixed j and k and varying $p = 1, \ldots 3$

$e_{ijk} x_k A_{kp}$ is a vector and we may apply the divergence theorem yielding

$$\int_{P} e_{ijk} \frac{\partial}{\partial x_p} \left(x_j A_{kp} \right) dV = 0$$

AND THEN

$$\int_P e_{ijk} \left[\delta_{jp} A_{kp} + x_j \frac{\partial A_{kp}}{\partial x_p} \right] dV = 0$$

As $\operatorname{div} A = 0$ $\left(\frac{\partial A_{ij}}{\partial x_j} = 0 \right)$

THEN WE HAVE $\int_P e_{ijk} A_{kj} \, dV = 0$

SINCE IT HOLDS FOR ALL $P \subset D$ WE MAY APPLY THE DIVERGENCE THEOM. AND

$$e_{ijk} A_{kj} = 0 \quad \forall x \in D$$

FINALLY, MULTIPLYING BOTH SIDES BY e_{ipq} AND USING THE IDENTITY $e_{ipq} e_{ijk} = \delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}$

$$(\delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}) A_{kj} = A_{qp} - A_{pq} = 0$$

$$\Longrightarrow \quad A = A^T$$