CHAPTER 1

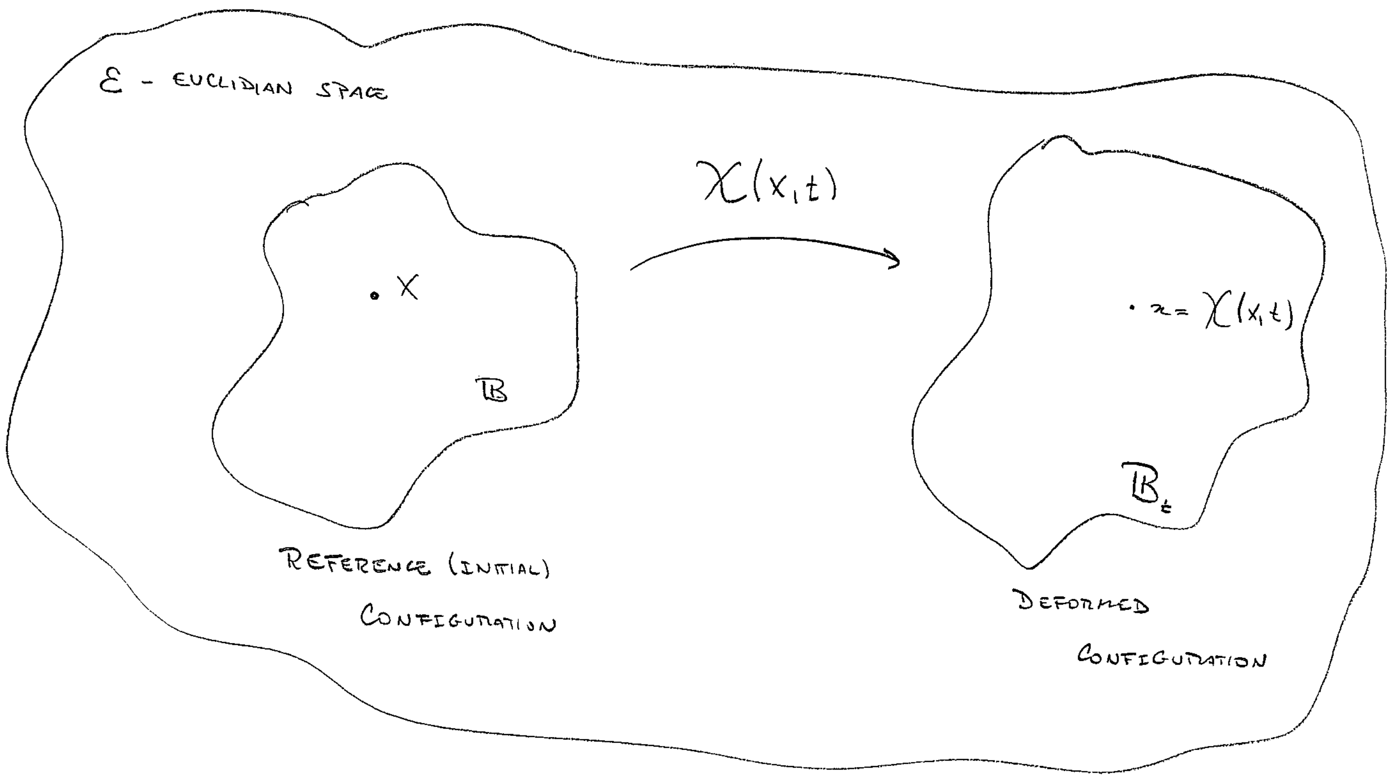
MATHEMATICAL PRELIMINAIRES

1.1) POINST. TENSORS. VECTORS

MECHANICAL RELATED FIELDS :

- SCALARS .... ρ (DENSITY)
- VECTORS .... $\underline{v}$ (VELOCITY)
- TENSORS .... T (STRESS)

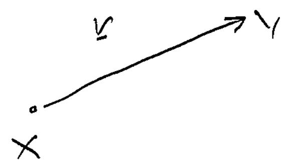
THE "BIG PICTURE"



NOTATION : UPPERCASE LETTERS DENOTE TENSORS (LINEAR TRANSFORMATIONS). LOWERCASE ONES ARE RESERVED FOR SCALARS AND VECTORS. UNDERBARS WILL BE EMPLOYED (e.g. $\underline{v}$) WHENEVER THE CONTEXT ITSELF IS NOT ABLE TO LEAD THE DISTINCTION BETWEEN SCALARS AND VECTORS. MOREOVER, VERY OFTEN TENSORS (DUE TO ITS REPRESENTATION IN COORDINATES) ARE ASSOCIATED TO MATRICES, IN THOSE CASES [T] WILL BE EMPLOYED. MORE IS TO COME WHEN WE TALK ABOUT COORDINATES.

LET $\mathbb{V}$ THE SPACE OF

$$\underline{v} = \underbrace{Y - X}_{\text{POINTS OF } \mathcal{E}}$$



AND ENDOWED WITH AN INNER PRODUCT DESIGNATED BY $\underline{u} \cdot \underline{v}$, WHICH IS DIRECTLY TIED TO THE INTUITIVE NOTION OF DISTANCE THROUGH THE RELATION

$$d = \sqrt{\underline{v}^2} = \sqrt{\underline{v} \cdot \underline{v}}$$

- POINTS, VECTORS AND TENSORS HAVE INTRINSIC VALUE (OFTEN IT CONDUCTS TO PHYSICAL INTERPRETATIONS, e.g.: VELOCITY) BUT THE REQUIRE TO BE EXPRESSED OF COORDINATE SYSTEMS.

- A CARTESIAN COORDINATE FRAME CONSISTS OF AN ORTHONORMAL BASIS $\{\underline{e}_i\} = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$. THEREFORE, THE (CARTESIAN) COMPONENTS (OR COORDINATES) OF VECTOR $\underline{u}$ ARE GIVEN BY

$$u_i = \underline{u} \cdot \underline{e}_i \qquad (i = 1, 2, 3)$$

SO THAT,

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{3} u_i v_i$$

(INDEED, IT IS VERY USUAL THE NOTATION $u_i v_i$)

SIMILARLY, THE COORDINATES OF A POINT X ARE

$$X_i = \underbrace{(X - O)}_{\text{THE "ORIGIN"}} \cdot \underline{e}_i$$

VECTOR $\underline{v}$ IS A LINEAR COMBINATION OF $\underline{u}$ AND $\underline{w}$ IF THERE EXISTS TWO SCALARS, α AND β, SUCH THAT

$$\underline{v} = \alpha \underline{u} + \beta \underline{w}$$

NOTE: THROUGH THE ABOVE DEFINITION WE CAN DEFINE A SUBSPACE (ARE YOU FAMILIAR WITH THIS CONCEPT?) DEPARTING FROM A SET OF VECTORS.

IN THE PRESENT CONTEXT, WE USE THE TERM TENSOR AS A SYNONYM FOR "LINEAR TRANSFORMATION FROM $\mathbb{V}$ TO $\mathbb{V}$.

THUS A TENSOR S IS A LINEAR MAP THAT ASSIGNS TO EACH VECTOR $\underline{u} \in \mathbb{V}$ A VECTOR

$$\underline{v} = S \underline{u}$$

AND

$$(S+T)\underline{u} = S\underline{u} + T\underline{u}$$

$$(\alpha S)\underline{u} = \alpha (S\underline{u}) \qquad (\alpha \in \mathbb{R})$$

$$\boxed{\begin{array}{l}\text{REMEMBER: LINEAR TRANSFORMATION} \\ S(\alpha \underline{u} + \beta \underline{v}) = \alpha S\underline{u} + \beta S\underline{v}\end{array}}$$

THE IDENTITY IS DEFINED BY

$$\mathbf{I}\,\underline{v} = \underline{v} \qquad \forall\, \underline{v} \in \mathbb{V}$$

THE PRODUCT OF TWO TENSORS S AND T (DENOTED AS ST) IS GIVEN BY

$$ST = S \circ T$$

OR BETTER

$$(ST)\,\underline{u} = S(T\underline{u})$$

IMPORTANT: GENERALLY $ST \neq TS$!

THE TRANSPOSE OF A TENSOR S, DENOTED AS S^T, IS THE UNIQUE TENSOR WITH THE PROPERTY

$$S\underline{u} \cdot \underline{v} = \underline{u} \cdot S^T \underline{v}$$

THEN

$$(S+T)^T = S^T + T^T \qquad (i)$$

$$(ST)^T = T^T S^T \qquad (ii)$$

$$(S^T)^T = S \qquad (iii)$$

EXERCISE: PROVE (i), (ii) AND (iii)

LET'S DO THE (ii)

BY DEFINITION

$$(ST)\underline{u} \cdot \underline{v} = \underline{u}\,(ST)^T \underline{v}$$

AND

$$(ST)\underline{u} \cdot \underline{v} = S(T\underline{u}) \cdot \underline{v} = T\underline{u} \cdot S^T\underline{v} =$$

$$= \underline{u} \cdot T^T S^T \underline{v}$$

$\boxed{\text{TRY THE OTHER TWO!!}}$

A TENSOR S IS SYMMETRIC IF

$$S = S^T$$

AND SKEW IF $S = -S^T$

THUS, EVERY TENSOR S CAN BE EXPRESSED IN AN UNIQUE WAY AS

$$S = E + W$$

WHERE

(SYMMETRIC PART) $E = \frac{1}{2}(S + S^T)$

(SKEW PART) $W = \frac{1}{2}(S - S^T)$

<u>DEFINITION</u>: TENSOR PRODUCT $\underline{a} \otimes \underline{b}$ IS SUCH THAT

$$(\underline{a} \otimes \underline{b})\,\underline{u} = (\underline{b} \cdot \underline{u})\,\underline{a}$$

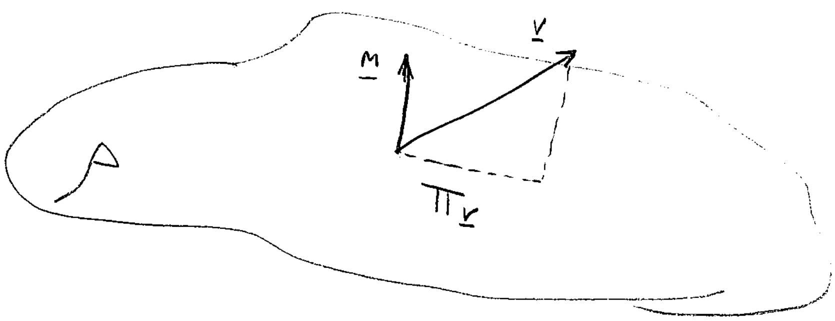
THE PROPERTIES BELLOW FOLLOW FROM THE DEFINITION OF THE TENSOR PRODUCT

$$(\underline{a} \otimes \underline{b})^T = (\underline{b} \otimes \underline{a})$$

$$(\underline{a} \otimes \underline{b})(\underline{c} \otimes \underline{d}) = (\underline{b} \cdot \underline{c})(\underline{a} \otimes \underline{d})$$

$$\sum_{i=1}^{3} \underline{e_i} \otimes \underline{e_i} = \underbrace{\underline{I}}_{\text{IDENTITY}}$$

A GEOMETRIC EXAMPLE INVOLVING THE TENSOR PRODUCT IS THE "PROJECTION OPERATOR" Π, WHICH PROJECTS VECTORS ONTO A GIVEN PLANE $\mathcal{P}$ (CHARACTERIZED THROUGH ITS NORMAL $\underline{n}$)



SUCH THAT

$$\Pi \underline{v} = \underline{v} - (\underline{v} \cdot \underline{n})\, \underline{n} = (\underline{I} - (\underline{n} \otimes \underline{n}))\, \underline{v}$$

THE COMPONENTS OF A TENSOR S ARE DEFINED BY

$$S_{ij} = \underline{e}_i \cdot S\,\underline{e}_j \qquad (i, j = 1, 3)$$

WHICH CAN BE ORGANIZED AS A MATRIX

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

THUS, "APPLYING" A TENSOR S TO A VECTOR $\underline{u}$ IS EQUIVALENT TO OBTAIN THE COMPONENTS OF THE RESULTING VECTOR $\underline{v}$ AS:

$$v_i = \sum_{j=1}^{3} S_{ij}\, u_j \qquad (i = 1, 3)$$

OR, SYMPLY (BY USING A CONVENTION IN WHICH THE SUMMATION IS IMPLIED OVER A REPEATED SUBSCRIPT)

$$v_i = S_{ij}\, u_j$$

j REPEATED SUBSCRIPT

EXAMPLES :

(i) IF $[A]$ AND $[B]$ ARE $m \times m$ SQUARE MATRICES AND $\underline{u}$; $\underline{v}$ AND $\underline{w}$ ARE VECTORS OF DIMENSION m, EXPRESS THE COMPONENTS OF $\underline{w}$ AS THE RESULTS OF SCALARS EQUATIONS:

$$\underline{w} = A\,\underline{v} + B\,\underline{u}$$

P.S: NOTE THAT ABOVE TENSORS AND MATRICES (OR OBJECTS AND THEIRS REPRESENTATIONS) ARE USED IN AN INDISTINCT FORM.

$$w_i = A_{ij}\, v_j + B_{ij}\, u_j$$

$$(i = 1, 2, 3)$$

(ii) FROM (i)

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

REMEMBER: $\underline{w} = I\,\underline{w}$

(iii) OBTAIN THE COMPONENTES OF TENSOR $S = TU$ AS FUNCTIONS OF THOSE OF T AND U

$$S_{ij} = T_{ik} U_{kj}$$

(iv) OBTAIN THE COORDINATES OF A TENSOR $S = (\underline{a} \otimes \underline{b})$ AS FUNCTIONS OF THE COMPONENTS OF VECTORS $\underline{a}$ AND $\underline{b}$

$$S_{ij} = a_i b_j$$

(JUST REMEMBER $S_{ij} = \underline{e_i} \cdot S \underline{e_j}$, FOLLOWING THE SAME LINES ONE ACHIEVES THAT ANY TENSOR T CAN BE PHRASED AS $T = T_{ij} \, \underline{e_i} \otimes e_j$)

THE OPERATOR TRACE IS DEFINED AS

$$tr(S) = \sum_i S_{ii}$$

WHICH ALLOWS THE INTRODUCTION OF AN INNER PRODUCT OF TENSORS, DEFINED AS

$$S . T = tr(S^T T) = S_{ij} T_{ij}$$

EXERCISE: PROVE THAT $\mathbb{I} \cdot S = \mathrm{tr}(S)$

THE DETERMINANT OF A TENSOR S IS THE DETERMINANT OF $[S]$ (IMPORTANT: REMEMBER THAT THE ENTRIES S_{ij} DEPEND ON THE BASIS $\{\underline{e}_i\}$. BUT, ANYWAY, THE DETERMINANT REMAINS THE SAME, INDEPENDENTLY OF THE CHOICE OF THE BASIS, THAT'S WHY IT IS CALLED AN INVARIANT)

NOTE: THAT IS A GOOD MOMENT TO RAISE AN IMPORTANT ISSUE: THE REPRESENTATION OF A TENSOR IN DIFFERENT COORDINATE FRAMES.

THE FIRST ASPECT TOWARD THIS ISSUE RELIES ON THE DEFINITION OF ORTHOGONAL TENSORS Q WHICH PRESERVES THE INNER PRODUCT

$$Q\underline{v} \cdot Q\underline{u} = \underline{v} \cdot \underline{u} \quad \forall \ \underline{v} \text{ AND } \underline{u} \in \mathbb{V}$$

THAT IMPLIES:

$$\underline{v} \cdot \underbrace{Q^T Q}\, \underline{u} = \underline{v} \cdot \underline{u}$$

$$\Downarrow$$

$$Q^T Q = I$$

CONSIDER NOW TWO ORTHONORMAL BASES, NAMELY $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ AND $\{\underline{e}_1^*, \underline{e}_2^*, \underline{e}_3^*\}$. SINCE BOTH FORM BASES, ANY VECTOR, AND IN PARTICULAR THE VECTORS $\underline{e}_i^*$ CAN BE REPRESENTED AS A LINEAR COMBINATION OF THE BASIS VECTORS $\underline{e}_1$, $\underline{e}_2$ AND $\underline{e}_3$.

THEREFORE

$$\underline{e}_i^* = Q_{ij}\, \underline{e}_j \qquad (i = 1, .., 3)$$

↳ 3 SCALARS FOR EACH $\underline{e}_i^*$

IT IS STRAIGHTFOWARD TO SEE THAT:

$$Q_{ij} = \underline{e}_i^* \cdot \underline{e}_j$$

AND, THEREFORE THE Q_{ij} $(i, j = 1, 3)$ CAN BE INTERPRETED AS A MATRIX SUCH THAT

$$\underline{e}_i^* = Q\, \underline{e}_i$$

EXERCISE: PLAYING WITH THE ABOVE RELATIONS, FIND

THAT

$$\underline{e}_i = Q^T \underline{e}_i^*$$

AND, AS A CONSEQUENCE

$$Q^T Q = Q Q^T = I$$

WHICH MEANS THAT Q IS AN ORTHONORMAL LINEAR TRANSFORMATION (TENSOR)

<u>NOTE</u>: FROM A GEOMETRICAL STANDPOINT, Q CAN BE INTERPRETED AS A ROTATION (PLEASE NOTE THAT THE 3 SCALARS Q_{ij}, FOR FIXED i, ARE DE COSINE OF THE ANGLE BETWEEN $\underline{e}_i^*$ AND $\underline{e}_j$)

THUS, WE CAN NOW RELATE THE <u>DIFFERENT</u> COMPONENTS OF THE VECTOR $\underline{V}$ IN TWO BASES. LET v_i AND v_i^* BE THE ith COMPONENT OF VECTOR $\underline{V}$ IN THE BASES $\{\underline{e}_i\}$ AND $\{\underline{e}_i^*\}$ THEN

$$v_i^* = Q_{ij} v_j \quad \text{OR, EQUIVALENTLY}$$

$$\{v'\} = [Q]\{v\}$$

THIS NOTATION CORRESPONDS TO ORGANIZING THE v_i^* COMPONENTS IN A VECTOR FORM

NOTE: IF $\underline{v}$ HAS A PHYSICAL MEAN, v_i ARE ITS COMPONENTS IN A SPECIFIC FRAME COORDINATE.

SIMILARLY, WE MAY RELATE THE DIFFERENT COMPONENTS OF A SINGLE TENSOR S IN TWO BASES. LET S_{ij} AND S_{ij}^* BE THE ij-COMPONENTS OF THE SAME TENSOR S IN THE BASES $\{\underline{e}_i\}$ AND $\{\underline{e}_i^*\}$ THEN

$$S^*_{ij} = Q_{ip} Q_{jq} S_{pq}$$

OR $$[S^*] = [Q][S][Q^T]$$

A TENSOR S IS INVERTIBLE IF THERE EXISTS A SECOND TENSOR S^{-1}, CALLED THE INVERSE OF S, SUCH THAT

$$S S^{-1} = S^{-1} S = I$$

IMPORTANT: S IS INVERTIBLE $\Longleftrightarrow \det S \neq 0$

THE FOLLOWING IDENTITIES WILL BE USEFUL AND PRESENTED WITHOUT PROOF

$$\det(ST) = (\det S)\,(\det T)$$

$$\det S^T = \det S$$

$$\det(S^{-1}) = \frac{1}{\det S}$$

$$(ST)^{-1} = T^{-1} S^{-1}$$

$$(S^{-1})^T = (S^T)^{-1} \underbrace{=}_{\text{NOTATION}} S^{-T}$$

1.2) EINGENVALUES AND EINGENVECTORS

THE MOST IMPORTANT RESULTS OF THE PRESENT ITEM ARE PHRASED IN 3 THEOREMS, NAMELY: SPECTRAL THEOREM, CAYLEY-HAMILTON THEOREM AND POLAR DECOMPOSITION THEOREM.

A SCALAR λ IS AN EINGENVALUE OF A TENSOR S IF THERE EXISTS A VECTOR $\underline{e}$ SUCH THAT

$$S\underline{e} = \lambda \underline{e}$$

IN THAT CASE $\underline{e}$ IS THE EINGENVECTOR. INDEED, IF $\underline{e}$ IS AN EINGENVECTOR THEN $\alpha \underline{e}$ $(\forall \alpha \in \mathbb{R})$ ARE TOO.

THE SPECTRUM OF S IS THE LIST $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ (USUALLY ORDERED SUCH THAT $\lambda_1 \leq \lambda_2 \ldots \leq \lambda_m$) ARE THE EINGENVALUES OF S.

NOTE: AN EINGENVALUE HAS MULTIPLICITY m IF THE EINGENVALUE EQUATION ABOVE IS TRUE FOR m NON PARALLEL VECTORS.

SPECTRAL THEOREM: LET S BE A SYMMETRIC TENSOR.

THEN THERE IS AN ORTHONORMAL BASIS FOR V (REMEMBER $S: V \to V$) CONSISTING ENTIRELY OF EINGENVECTORS OF S. THAT MEANS

$$S = \sum_{i} \lambda_i \left(\underline{e}_i \otimes \underline{e}_i \right)$$

THUS, IN THAT CASE

$$[S] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

A DIAGONAL MATRIX ←

POLAR DECOMPOSITION THEOREM : LET $F \in Lin^+$ (THE SET OF ALL TENSORS S WITH $\det S > 0$; WE WILL SOON THAT THIS PROPERTY PLAYS AN IMPORTANT ROLE IN THE MECHANICS OF DEFORMABLE SOLIDS!!). THEN THERE EXIST POSITIVE DEFINITE (MEANING THAT $U\underline{v}\cdot\underline{v} > 0 \ \forall \underline{v} \neq 0$) AND SYMMETRIC TENSORS U AND V AND PLUS A ROTATION R SUCH THAT

$$F = RU = VR$$

WHERE

$$U = (F^T F)^{1/2} \quad \text{AND} \quad V = (F F^T)^{1/2}$$

WHERE

$$U = \sum_i \sqrt{\lambda_i}\ \underline{e_i} \otimes \underline{e_i}$$

(THE RESPECTIVE RESULT HOLDS FOR V)

(λ_i AND $\underline{e_i}$ ARE, RESPECTIVELY, EIGENVALUE AND EINGENVECTOR OF $F^T F$)

THE REPRESENTATION $F = RU$ IS THE RIGHT POLAR DECOMPOSITION OF F

QUESTION : DOES IT ALWAYS MAKE SENSE TALKING OF $\sqrt{\lambda_i}$ AS A REAL NUMBER IN THE PREVIOUS THEOREM?

THE ANSWER IS YES.

FACT 1: $F^T F$ IS A POSITIVE NON NEGATIVE TENSOR,

$$F^T F \underline{v} \cdot \underline{v} \geq 0$$

(NOTE $F\underline{v} \cdot F\underline{v} > 0$ UNLESS IT IS 0, WHICH IMPLIES $F\underline{v} = \underline{0} \rightarrow \lambda = 0 \rightarrow \sqrt{\lambda} = 0$)

FACT 2 : AS $F^T F$ IS SYMMETRIC $\left((F^T F)^T = F^T F\right)$

THEN THE SPECTRAL THEOREM APPLIES, SO

$$F^T F = \sum_1^3 \lambda_i \, \underline{e_i} \otimes \underline{e_i}$$

COMBINING (FACT 1) AND (FACT 2) WE CONCLUDE THAT $\lambda_i \geq 0$, THEREFORE $\sqrt{\lambda_i} \in \mathbb{R}$

CAYLEY-HAMILTON THEOREM: EVERY TENSOR S SATISFIES ITS OWN CHARACTERISTIC EQUATION

$$S^3 - i_1(S)\, S^2 + i_2(S)\, S - i_3(S)\, I = 0$$

WHERE $i_j(S)$ ARE CALLED AI INVARIANTS OF S (THIS NAME IS AFTER THE FACT THAT $i_s(S)$ DOES NOT THE COORDINATE FRAME USED TO REPRESENT S) AND THEY ARE DEFINED AS

$$i_1(S) = \mathrm{tr}\, S$$

$$i_2(S) = \frac{1}{2}\left[(\mathrm{tr}\, S)^2 - \mathrm{tr}(S^2)\right]$$

$$i_3(S) = \det S$$

THE ABOVE THEOREM HAS AN IMPORTANCE CONSEQUENCE IN ORDER TO HAVE A CHARACTERIZATION OF S THROUGH ITS INVARIANTS.

1.3) Advanced Examples and Miscellaneous

The examples bellow are intentionally not in the same order the subjects have been presented in the last classes. Besides, as much as possible each example contains elements of different subjects.

(1) We introduced in the last lesson the definition of an orthogonal tensor through the relation

$$Q^T Q = Q Q^T = I$$

$$\left(\text{which implies } Q = Q^{-1}\right)$$

As an important property, orth. tensors preserves the norm of a vetor: $\underline{u} = Q\,\underline{v} \longrightarrow \|\underline{u}\| = \|\underline{v}\|$.

Dem.:

$$\|\underline{u}\|^2 = \underline{u}\cdot\underline{u} = Q\underline{v}\cdot Q\underline{v} \underbrace{=}_{\substack{\text{By the}\\\text{definition of}\\\text{transpose}}} \underline{v}\cdot Q^T Q\,\underline{v} \overset{\substack{\text{See the def.}\\\text{above}}}{=} \underline{v}\cdot\underline{v} = \|\underline{v}\|^2$$

(2) DEPARTING FROM THE DEFINITION OF INNER PRODUCT FOR TENSORS (REMEMBER $S \cdot T = tr(S^T T)$), PROOF THE FOLLOWING RELATIONS

(i) $I \cdot S = tr(S)$

(ii) $\underline{u} \cdot S\underline{v} = S \cdot (\underline{u} \otimes \underline{v})$

Sol.

(i) $I \cdot S = tr(\underset{\text{IDENTITY}}{\underbrace{I^T}} S) = tr(S)$

(ii) $S \cdot (\underline{u} \otimes \underline{v}) = tr(S^T(\underline{u} \otimes \underline{v}))$

LET ME PROPOSE TWO DIFFERENT PROOFS (THE IDEA IS TO EXPLORE MORE CONCEPTS FROM THE EXERCISE). BOTH SOLUTIONS WILL USE (IN A CERTAIN MOMENT) THE TENSORS ON VECTORS EXPRESSED IN COMPONENTS

<u>Sol</u>. 1: (REMEMBER FROM PAGE 11 OF THE NOTES)

$$tr\left(S^T(\underline{u}\otimes\underline{v})\right) = S_{ij}\,(\underline{u}\otimes\underline{v})_{ij} =$$

BUT (FROM THE SAME PAGE OF THE NOTES)

$$= S_{ij}\,u_i v_j = u_i\,S_{ij} v_j =$$

$$= u_i\,(S\underline{v})_i = \underline{u}\cdot S\underline{v}$$

<u>Sol. 2</u>:

$$S\cdot(\underline{u}\otimes\underline{v}) = tr\left(S^T(\underline{u}\otimes\underline{v})\right) = tr\left((S^T\underline{u}\otimes\underline{v})\right) =$$

(WHY ???)

$$= (S^T_{ij}\,u_j)\,v_i = u_j\,S^T_{ij} v_i = u_j\,S_{ji} v_i =$$

$$= u_j\,(S\underline{v})_j = \underline{u}\cdot S\underline{v}$$

JUST PLAYING WITH INDEXES

③ PROOF THAT

(i) $S(\underline{a} \otimes \underline{b}) = (S\underline{a}) \otimes \underline{b}$

(ii) $\sum_{i=1}^{3} ((S\underline{e_i}) \otimes \underline{e_i} = S$

NOTE: IT IS IMPORTANT TO REMARK THA WE USED (i) IN THE PREVIOUS EXAMPLE

Sol

(i)

$$\forall \underline{u} \quad [S(\underline{a} \otimes \underline{b})]\underline{u} = S((\underline{u} \cdot \underline{b})\underline{a}) = (\underline{u} \cdot \underline{b}) S\underline{a} =$$

$$= (S\underline{a} \otimes \underline{b})$$

(ii) FROM (i)

$$\sum_i (S\underline{e_i}) \otimes \underline{e_i} = \sum_i S(\underline{e_i} \otimes \underline{e_i}) = S \underbrace{\sum_i e_i \otimes e_i}_{I} = S$$

4 DETERMINE THE EINGENVALUES AND EIGENVECTORS FOR $A = \alpha I + \beta (\underline{m} \otimes \underline{m})$

WHERE $\alpha, \beta \in \mathbb{R}$ AND $\underline{m}$ IS AN UNITY VECTOR $(\|\underline{m}\| = 1)$

$$A \underline{u} = \lambda \underline{u} \qquad (\|\underline{u}\| = 1 \ (\text{HOW?}))$$

$$(\alpha I + \beta \, \underline{m} \otimes \underline{m}) \underline{u} = \lambda \underline{u}$$

$$\alpha \underline{u} + \beta (\underline{m} \cdot \underline{u}) \underline{m} = \lambda \underline{u}$$

THUS $\underline{m} = \underline{u}$ (ONCE AGAIN, WHY?)
(OR $\underline{m} \cdot \underline{u} = 0$)

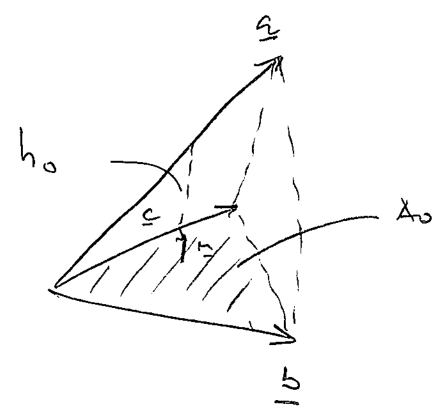
SO
$$(\alpha + \beta) \underline{u} = \lambda \underline{u}$$

THUS $\lambda = \alpha + \beta$ IS AN EIGENVALUE OF A CORRESPONDING TO THE EINGENVECTOR $\underline{m}$. MOREOVER, BUILD AN ORTHONORMAL BASIS UPON $\underline{m}$ $(\{\underline{m}, \underline{u}, \underline{\ell}\})$ IS EASY TO VERIFY THAT α IS AN EINGENVALUE WITH MULTIPLICITY 2 HAVING AS EIGENVECTORS $\underline{u}$ AND $\underline{\ell}$

- PLEASE NOTE THAT WE COULD HAVE FOLLOWED A DIFFERENT PATH: JUST WRITE DOWN THE MATRIX CORRESPONDING TO A USING A VECTOR BASIS BUILT UPON $\underline{m}$.

⑤ INTERPRET THE QUANTITY $\underline{a} \cdot (\underline{b} \times \underline{c})$ GEOMETRICALLY. REMEMBER FROM YOUR LINEAR ALGEBRA BACKGROUND THE VECTOR PRODUCT $\underline{b} \times \underline{c}$, WHICH GIVES RISE TO A VECTOR $\underline{d} = \|\underline{b}\| \, \|\underline{c}\| \sin\theta \, \underline{m}$, WHERE θ STANDS FOR THE ANGLE FORMED BY THE TWO VECTORS AND $\underline{m}$ IS A UNITY VECTOR NORMAL TO THEIR PLANE.

$\underline{\text{SOL.}}$ ASSUMING THAT $\underline{a}$ IS NOT IN THE PLANE SPANNED BY $\underline{b}$ AND $\underline{c}$ (WHAT IF IT WAS?), THE 3 VECTORS FORM A TETRAHEDRON:



ITS VOLUME, $V_0 = \frac{1}{3} A_0 h_0$ WHERE A_0 IS THE AREA OF THE BASIS (TRIANGLE FORMED BY $\underline{b}$ AND $\underline{c}$) AND

h_0 ITS HEIGHT (PLEASE NOTE THAT h_0 IS THE NORM OF A VECTOR PARALLEL TO $\underline{n}$)

THUS $V_0 = \frac{1}{3} \underbrace{\frac{\|\underline{b} \times \underline{c}\|}{2}}_{\text{AREA}} \cdot (\underline{a} \cdot \underline{n})$

BUT $\underline{n} = \dfrac{\underline{b} \times \underline{c}}{\|\underline{b} \times \underline{c}\|}$

SO $V_0 = \frac{1}{6} \underline{a} \cdot (\underline{b} \times \underline{c})$

6) CONSIDER A SYMMETRIC POSITIVE DEFINITE TENSOR S. SHOW THAT IT HAS A UNIQUE SYMMETRIC POSITIVE SQUARE ROOT, I.E, SHOW THAT THERE IS A UNIQUE SYMMETRIC POSITIVE DEFINITE LINEAR TENSOR T FOR WHICH $T^2 = S$

SOL :

FROM THE SPECTRAL THEOREM COMBINED WITH S BEING POSITIVE DEFINITE, THE THREE EINGENVALUES $(\sigma_1, \sigma_2, \sigma_3)$ ARE POSITIVE VALUED AND THE CORRESPONDING EINGENVECTORS $\{\underline{s}_1, \underline{s}_2, \underline{s}_3\}$ MAY BE TAKEN ORTHONORMAL.

THUS

$$S = \sum_{i=1}^{3} \sigma_i \left(\underline{s}_i \otimes \underline{s}_i \right)$$

SO, LET US DEFINE

$$T = \sum_{i=1}^{3} \sqrt{\sigma_i} \left(\underline{s}_i \otimes \underline{s}_i \right)$$

COMES IMMEDIATELLY THAT T IS SYMMETRIC, POSITIVE DEFINITE AND $T^2 = S$ (BETTER YOU TRY TO PROOF ALL THE THREE CLAIMS!). SO FAR, WE'VE PROVED THE EXISTENCE OF A SQUARE-ROOT. WHAT REMAINS IS TO SHOW UNIQUENESS ... FOR THAT, SUPPOSE THAT S HAS TWO S.P.D (SYMMETRIC POSITIVE DEFINITE) ROOTS T_1 AND T_2 ($S = T_1^2$ AND $S = T_2^2$). LET $\sigma > 0$ AN $\underline{s}$ BE AN EINGENPAIR OF S. THEN $S\underline{s} = \sigma \underline{s}$ AND

$$T_1^2 \underline{s} = \sigma \underline{s}$$

THUS

$$(T_1 + \sqrt{\sigma}\, I)(T_1 - \sqrt{\sigma}\, I)\, \underline{s} = 0$$

(DID YOU REALLY GET THIS?)

BY SETTING $\underline{r} = (T_1 - \sqrt{\sigma}\, I)\, \underline{s}$

$$T_1 \underline{r} = -\sqrt{\sigma}\, \underline{r}$$

THUS EITHER $\underline{r} = 0$ OR $\underline{r}$ IS A NEGATIVE EIGENVECTOR $(-\sqrt{\sigma} < 0)$,

BUT AS T_1 IS P.D $\longrightarrow$ $\underline{r} = 0$ AND

$$T_1 \underline{s} = \sqrt{\sigma}\, \underline{s}$$

THE SAME HAPPENS TO T_2, $T_2 \underline{s} = \sqrt{\sigma}\, \underline{s}$

THUS $T_1 \underline{s} = T_2 \underline{s}$

THAT HOLDS FOR EACH ONE OF THE EIGENVECTORS OF S,

AS THEY FORM A BASIS (REMEMBER THE SPECTRAL THEOREM)

IT IMPLIES THAT $T_1 \underline{u} = T_2 \underline{u} \quad \forall \underline{u}$

$\left(\text{AS } T_1 \underline{u} = T_1\left(\sum_i u_i \underline{s}_i\right) = \sum_i u_i T_1 s_i = \sum_i u_i T_2 s_i = T_2 \underline{u} \right)$ $\blacksquare$