MAE4700/5700
Finite Element Analysis for Mechanical and Aerospace Design

Cornell University, Fall 2009

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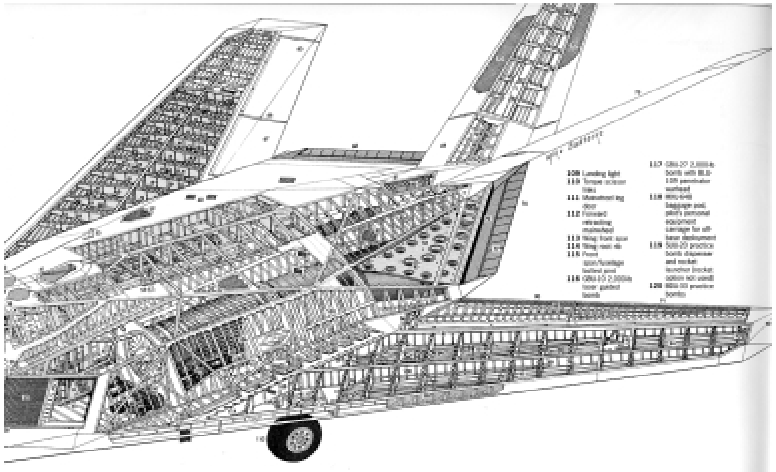
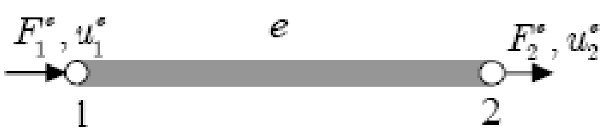
Truss analysis



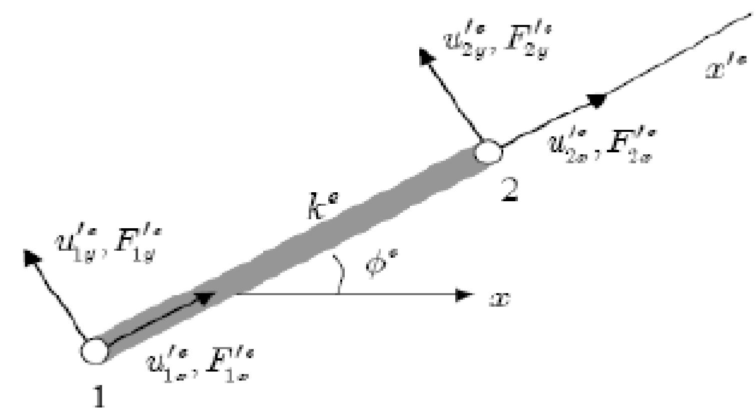
- The truss element forms the basic element of such structures.
- It can only take forces along its span (no moments).

- Many engineering structures consist of straight members connected at their ends by bolts, pins or welding.

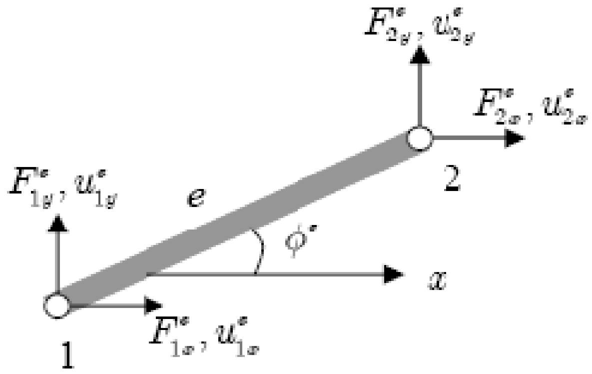


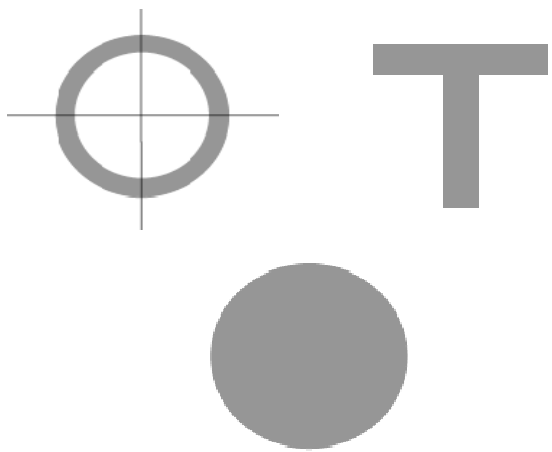
Truss analysis

- Internal forces in a truss element act along the member $F_{1y}^{'(e)} = F_{2y}^{'(e)} = 0$
- However, displacements at the nodes can have both components (x'- and y'-directions, in local coordinates).
- This is due to rotation



Truss analysis



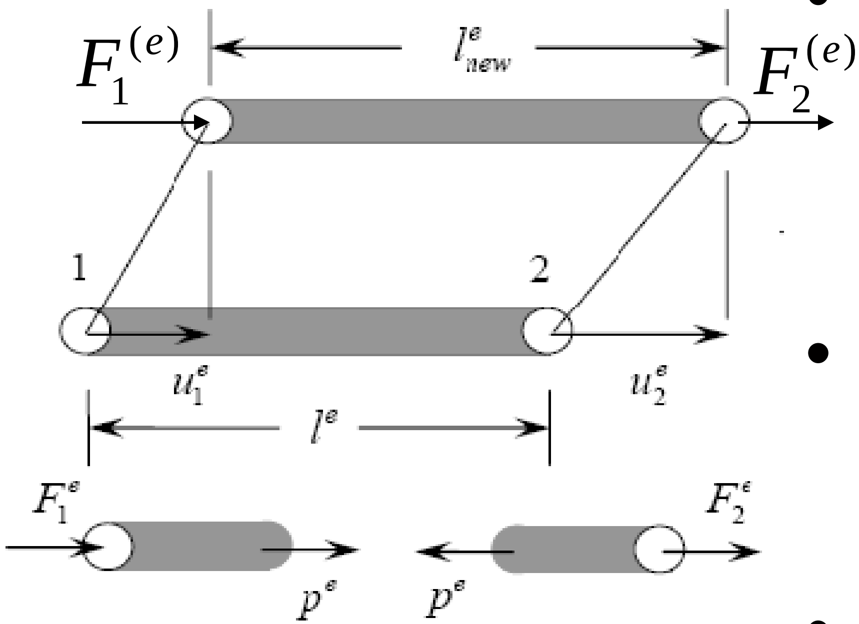


- To analyze a truss element in the global coordinates *x* and *y*, you need to account for both components of displacement:

$$u_{1x}^{(e)}, u_{1y}^{(e)}, u_{2x}^{(e)}, u_{2y}^{(e)}$$

- Also note that the cross section of truss elements can vary as shown.

Stiffness of a truss element



$$\begin{Bmatrix} F_1^{(e)} \\ F_2^{(e)} \end{Bmatrix} = \begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$$

$$k^e = \frac{A^e E^e}{L^e}$$

- The internal force p^e in the truss is given (see free body diagram) as:

$$p^e = F_2^{(e)} = -F_1^{(e)} = A^e \sigma^e$$

- Assuming elastic deformations:

$$p^e = F_2^{(e)} = -F_1^{(e)} = A^e E^e \varepsilon^e$$

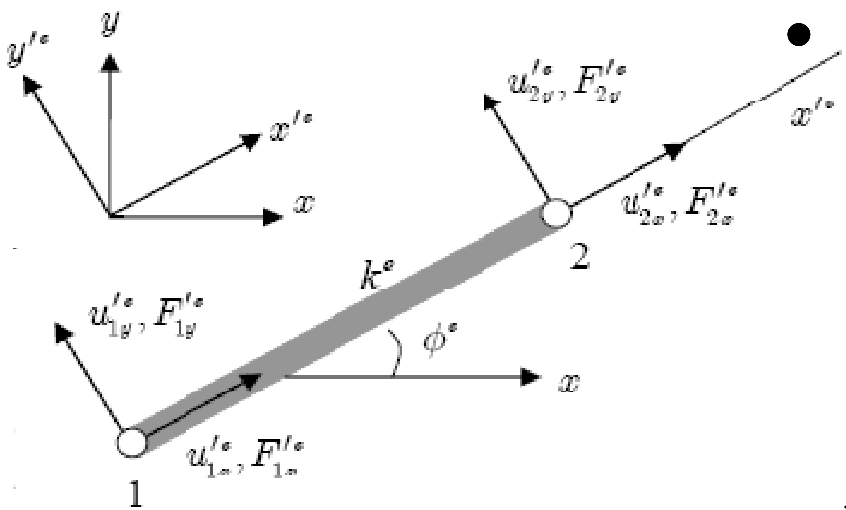
- The (small) strain is given as:

$$\varepsilon^e = \frac{u_2^e - u_1^e}{L^e}$$

- Finally: $F_2^{(e)} = -F_1^{(e)} = \dfrac{A^e E^e}{L^e}(u_2^e - u_1^e) =$

$$= k^e (u_2^e - u_1^e)$$

Truss element stiffness in local coordinates



$$\begin{Bmatrix} F_{1x}^{'(e)} \\ F_{2x}^{'(e)} \end{Bmatrix} = \begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u_{1x}^{'(e)} \\ u_{2x}^{'(e)} \end{Bmatrix}$$

- We can re-write the element stiffness equations as:

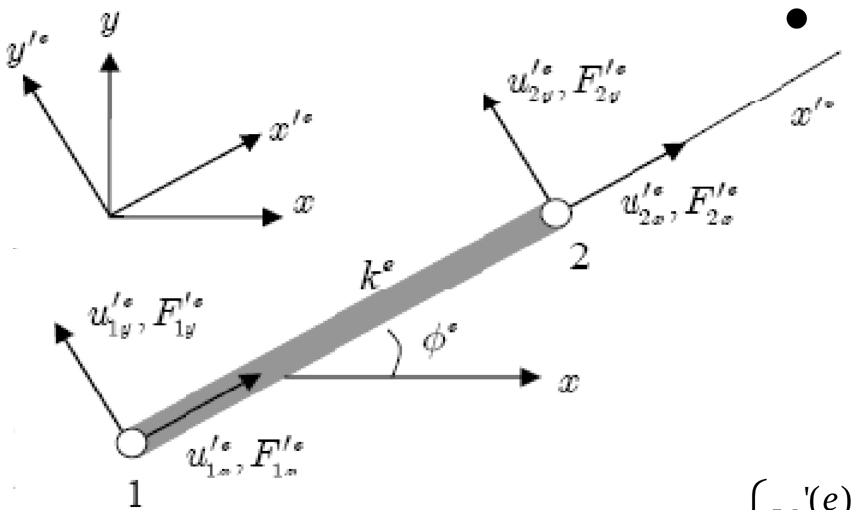
$$\underbrace{\begin{Bmatrix} F_{1x}^{'(e)} \\ F_{1y}^{'(e)} \\ F_{2x}^{'(e)} \\ F_{2y}^{'(e)} \end{Bmatrix}}_{\{F'^{(e)}\}} = k^{(e)} \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{[K'^{(e)}]} \underbrace{\begin{Bmatrix} u_{1x}^{'(e)} \\ u_{1y}^{'(e)} \\ u_{2x}^{'(e)} \\ u_{2y}^{'(e)} \end{Bmatrix}}_{\{d'^{(e)}\}}$$

Notice (as it should be) that:

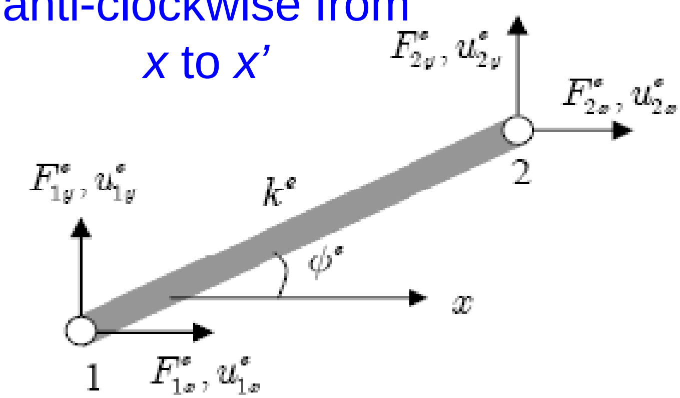
$$F_{1y}^{'(e)} = F_{2y}^{'(e)} = 0$$

$$\{F'^{(e)}\} = [K'^{(e)}]\{d'^{(e)}\}$$

Element stiffness in global coordinates



The angle ϕ^e is measured anti-clockwise from *x* to *x'*

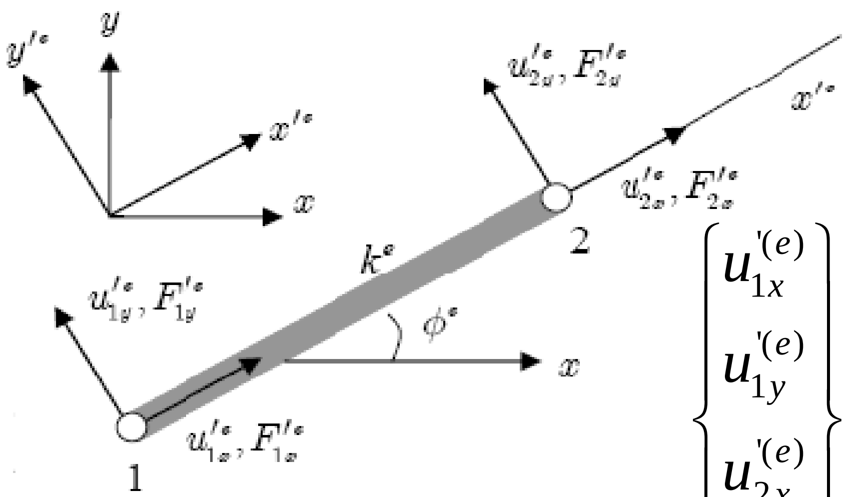


- We need to be able to transform displacements from the *x'* and *y'* axes to displacements along the *x* and *y* axes. We start with the reverse:

$$\underbrace{\begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{1y} \\ u'^{(e)}_{2x} \\ u'^{(e)}_{2y} \end{Bmatrix}}_{\{d'^{(e)}\}} = \underbrace{\begin{bmatrix} \cos\phi^e & \sin\phi^e & 0 & 0 \\ -\sin\phi^e & \cos\phi^e & 0 & 0 \\ 0 & 0 & \cos\phi^e & \sin\phi^e \\ 0 & 0 & -\sin\phi^e & \cos\phi^e \end{bmatrix}}_{T^{(e)}} \underbrace{\begin{Bmatrix} u^{(e)}_{1x} \\ u^{(e)}_{1y} \\ u^{(e)}_{2x} \\ u^{(e)}_{2y} \end{Bmatrix}}_{\{d^{(e)}\}}$$

Transformation matrix $T^{(e)}$

Coordinate transformation

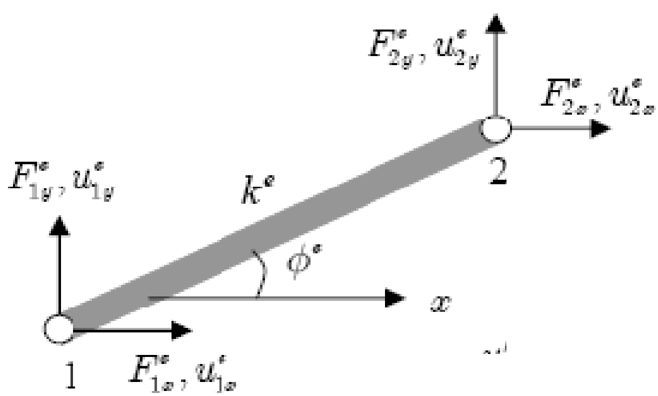


$$\{d'^{e}\} = [T^{e}]\{d^{e}\}$$

$$\underbrace{\begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{1y} \\ u'^{(e)}_{2x} \\ u'^{(e)}_{2y} \end{Bmatrix}}_{\{d'^{(e)}\}} = \underbrace{\begin{bmatrix} \cos\phi^e & \sin\phi^e & 0 & 0 \\ -\sin\phi^e & \cos\phi^e & 0 & 0 \\ 0 & 0 & \cos\phi^e & \sin\phi^e \\ 0 & 0 & -\sin\phi^e & \cos\phi^e \end{bmatrix}}_{[T^{(e)}]} \underbrace{\begin{Bmatrix} u^{(e)}_{1x} \\ u^{(e)}_{1y} \\ u^{(e)}_{2x} \\ u^{(e)}_{2y} \end{Bmatrix}}_{\{d^{(e)}\}}$$

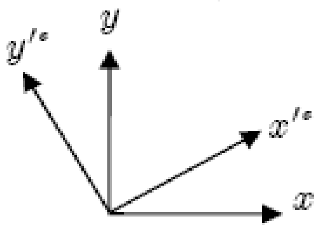
Note that :

$$[T^{e}]^{T}[T^{e}] = [I]$$



$$\boxed{\{d^{e}\} = [T^{e}]^{T}\{d'^{e}\}}$$

Coordinate transformation

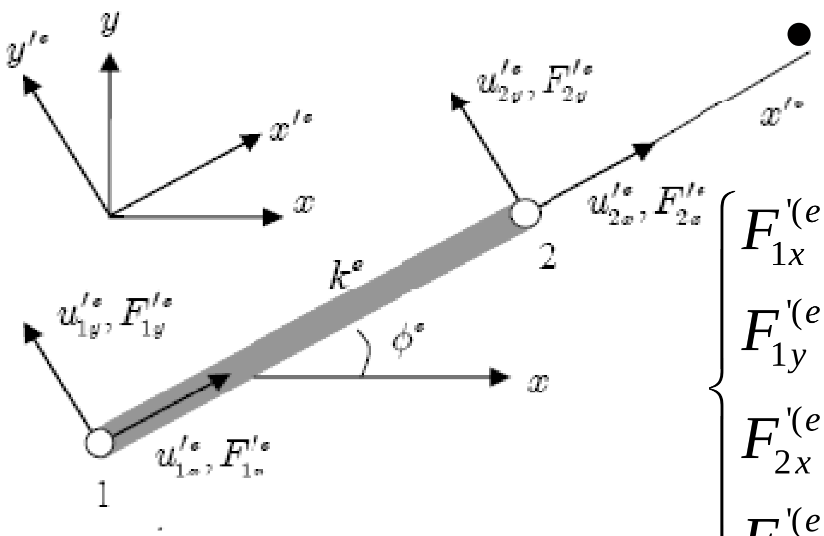


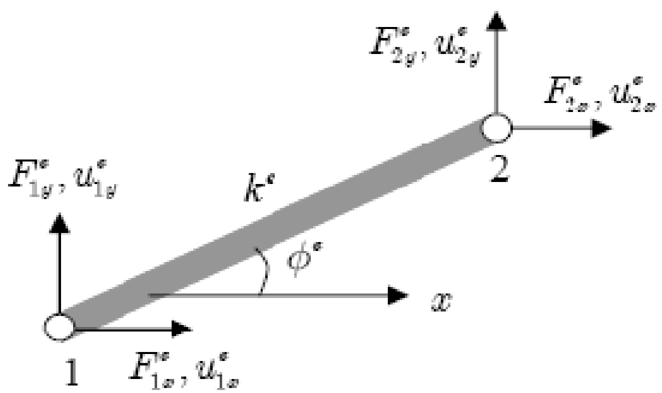
Verify that: $[T^e]^T[T^e]=[I]_{4x4}$

$$\underbrace{\begin{bmatrix} \cos\phi^e & -\sin\phi^e & 0 & 0 \\ \sin\phi^e & \cos\phi^e & 0 & 0 \\ 0 & 0 & \cos\phi^e & -\sin\phi^e \\ 0 & 0 & \sin\phi^e & \cos\phi^e \end{bmatrix}}_{T^{e^T}} \underbrace{\begin{bmatrix} \cos\phi^e & \sin\phi^e & 0 & 0 \\ -\sin\phi^e & \cos\phi^e & 0 & 0 \\ 0 & 0 & \cos\phi^e & \sin\phi^e \\ 0 & 0 & -\sin\phi^e & \cos\phi^e \end{bmatrix}}_{T^e} =$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I_{4\times 4}}$$

Stiffness of a truss element





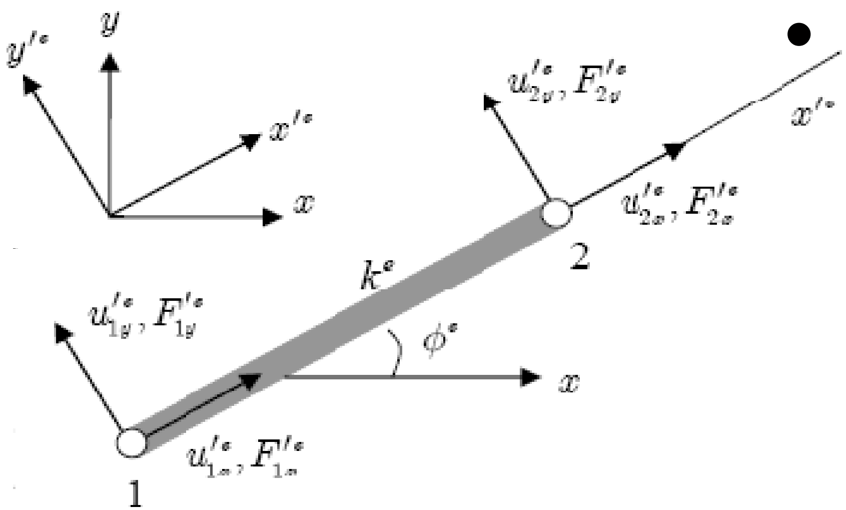
- Similarly for the forces:

$$\{F'^e\}=[T^e]\{F^e\}$$

$$\underbrace{\begin{Bmatrix} F'^{(e)}_{1x} \\ F'^{(e)}_{1y} \\ F'^{(e)}_{2x} \\ F'^{(e)}_{2y} \end{Bmatrix}}_{\{F'^e\}} = \underbrace{\begin{bmatrix} \cos\phi^e & \sin\phi^e & 0 & 0 \\ -\sin\phi^e & \cos\phi^e & 0 & 0 \\ 0 & 0 & \cos\phi^e & \sin\phi^e \\ 0 & 0 & -\sin\phi^e & \cos\phi^e \end{bmatrix}}_{[T^{(e)}]} \underbrace{\begin{Bmatrix} F^{(e)}_{1x} \\ F^{(e)}_{1y} \\ F^{(e)}_{2x} \\ F^{(e)}_{2y} \end{Bmatrix}}_{\{F^e\}}$$

$$\boxed{\{F^e\}=[T^e]^T\{F'^e\}}$$

Stiffness of a truss element



- Using $\{F^{'(e)}\} = [K^{'(e)}]\left\{d^{'(e)}\right\}$

  and the transformation equations, $\{F^{'(e)}\} = [T^{(e)}]\left\{F^{(e)}\right\}$

  $$\{d^{'(e)}\} = [T^{(e)}]\left\{d^{(e)}\right\}$$

  we can write the stiffness in the *x,y* system as follows:

$$\{F^{'(e)}\} = [K^{'(e)}]\left\{d^{'(e)}\right\} \Rightarrow$$

$$[T^{(e)}]\left\{F^{(e)}\right\} = [K^{'(e)}][T^{(e)}]\left\{d^{(e)}\right\} \Rightarrow \left\{F^{(e)}\right\} = \underbrace{[T^{(e)}]^T[K^{'(e)}][T^{(e)}]}_{[K^{(e)}]}\left\{d^{(e)}\right\} \Rightarrow$$

$$[K^{(e)}] = [T^{(e)}]^T[K^{'(e)}][T^{(e)}]$$

Truss element stiffness

$$[K^{(e)}] = [T^{(e)}]^T [K'^{(e)}][T^{(e)}]$$

$$[K'^{(e)}] = k^{(e)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [T^{(e)}] = \begin{bmatrix} \cos\phi^e & \sin\phi^e & 0 & 0 \\ -\sin\phi^e & \cos\phi^e & 0 & 0 \\ 0 & 0 & \cos\phi^e & \sin\phi^e \\ 0 & 0 & -\sin\phi^e & \cos\phi^e \end{bmatrix}$$

$$[K^{(e)}] = k^{(e)} \begin{bmatrix} \cos^2\phi^e & \sin\phi^e\cos\phi^e & -\cos^2\phi^e & -\sin\phi^e\cos\phi^e \\ \sin\phi^e\cos\phi^e & \sin^2\phi^e & -\sin\phi^e\cos\phi^e & -\sin^2\phi^e \\ -\cos^2\phi^e & -\sin\phi^e\cos\phi^e & \cos^2\phi^e & \sin\phi^e\cos\phi^e \\ -\sin\phi^e\cos\phi^e & -\sin^2\phi^e & \sin\phi^e\cos\phi^e & \sin^2\phi^e \end{bmatrix}$$

Truss element stiffness

$$\begin{Bmatrix} F_{1x}^{(e)} \\ F_{1y}^{(e)} \\ F_{2x}^{(e)} \\ F_{2y}^{(e)} \end{Bmatrix} = k^{(e)} \left[\begin{array}{cc:cc} \cos^2\phi^e & \sin\phi^e\cos\phi^e & -\cos^2\phi^e & -\sin\phi^e\cos\phi^e \\ \sin\phi^e\cos\phi^e & \sin^2\phi^e & -\sin\phi^e\cos\phi^e & -\sin^2\phi^e \\ \hdashline -\cos^2\phi^e & -\sin\phi^e\cos\phi^e & \cos^2\phi^e & \sin\phi^e\cos\phi^e \\ -\sin\phi^e\cos\phi^e & -\sin^2\phi^e & \sin\phi^e\cos\phi^e & \sin^2\phi^e \end{array}\right] \begin{Bmatrix} u_{1x}^{(e)} \\ u_{1x}^{(e)} \\ u_{2x}^{(e)} \\ u_{2y}^{(e)} \end{Bmatrix}$$

- Note the *2x2* symmetric submatrix structure
- This implies that you can reverse the numbering of nodes (1 and 2) without any changes in the element stiffness.

$$\begin{Bmatrix} F_{2x}^{(e)} \\ F_{2y}^{(e)} \\ F_{1x}^{(e)} \\ F_{1y}^{(e)} \end{Bmatrix} = k^{(e)} \begin{bmatrix} \cos^2\phi^e & \sin\phi^e\cos\phi^e & -\cos^2\phi^e & -\sin\phi^e\cos\phi^e \\ \sin\phi^e\cos\phi^e & \sin^2\phi^e & -\sin\phi^e\cos\phi^e & -\sin^2\phi^e \\ -\cos^2\phi^e & -\sin\phi^e\cos\phi^e & \cos^2\phi^e & \sin\phi^e\cos\phi^e \\ -\sin\phi^e\cos\phi^e & -\sin^2\phi^e & \sin\phi^e\cos\phi^e & \sin^2\phi^e \end{bmatrix} \begin{Bmatrix} u_{2x}^{(e)} \\ u_{2x}^{(e)} \\ u_{1x}^{(e)} \\ u_{1y}^{(e)} \end{Bmatrix}$$

Assembly process

- The assembly process is identical to the one discussed for `spring structures' and it will not be repeated here in its general form (no need to show at this point complicated looking matrix operations).
- We will however provide soon a simple example demonstrating this assembly process.

Generalizing the application of essential BCs

- You already have seen through an example how essential boundary conditions are applied to the global system of eqs: $[K]\{d\}=\{F\}$
- In essence, we partition the stiffness matrix in a way that separates known from unknown degrees of freedom as follows:

$$\begin{bmatrix} K_E & K_{EF} \\ K_{EF}^T & K_F \end{bmatrix} \begin{Bmatrix} \overline{d}_E \\ d_F \end{Bmatrix} = \begin{Bmatrix} \overline{f}_E \\ f_F \end{Bmatrix}$$

$\overline{d}_E$: Known displacements

d_F : Unknown displacements

f_F : Applied (known) forces

$\overline{f}_E$: Unknown reaction forces corresponding to nodes/directions with prescribed displacement

Generalizing the application of essential BCs

$$\begin{bmatrix} K_E & K_{EF} \\ K_{EF}^T & K_F \end{bmatrix} \begin{Bmatrix} \overline{d}_E \\ d_F \end{Bmatrix} = \begin{Bmatrix} \overline{f}_E \\ f_F \end{Bmatrix} \Rightarrow \qquad \begin{aligned} K_E \overline{d}_E + K_{EF} d_F &= \overline{f}_E \\ K_{EF}^T \overline{d}_E + K_F d_F &= f_F \end{aligned}$$

- The unknown displacements are obtained from the 2nd eq. as:

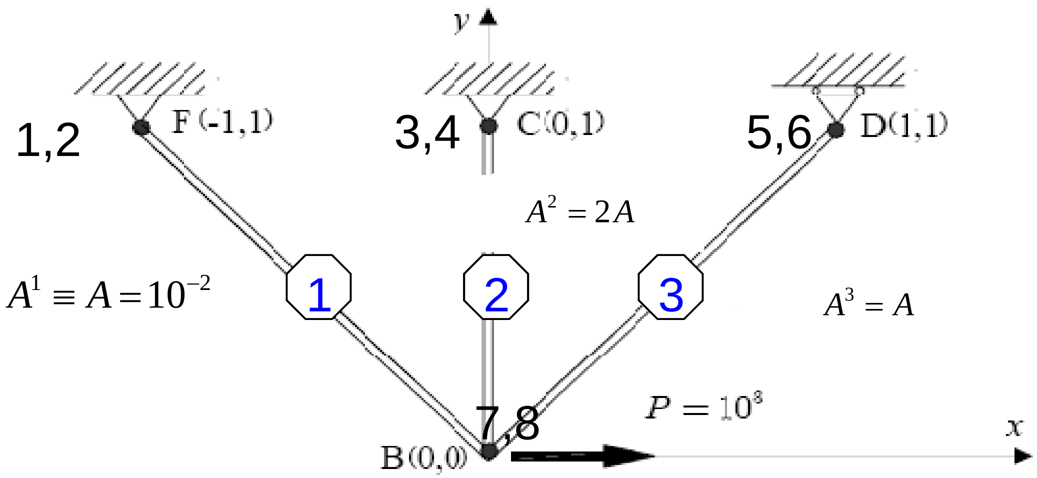
$$K_{EF}^T \overline{d}_E + K_F d_F = f_F \Rightarrow d_F = K_F^{-1}(f_F - K_{EF}^T \overline{d}_E)$$

- With known d_F , we can return to the 1st eq. to compute the reaction forces:

$$\overline{f}_E = K_E \overline{d}_E + K_{EF} d_F$$

- Note that the matrix K_F is symmetric and positive definite, so a solution for d_F always exists!

A truss example

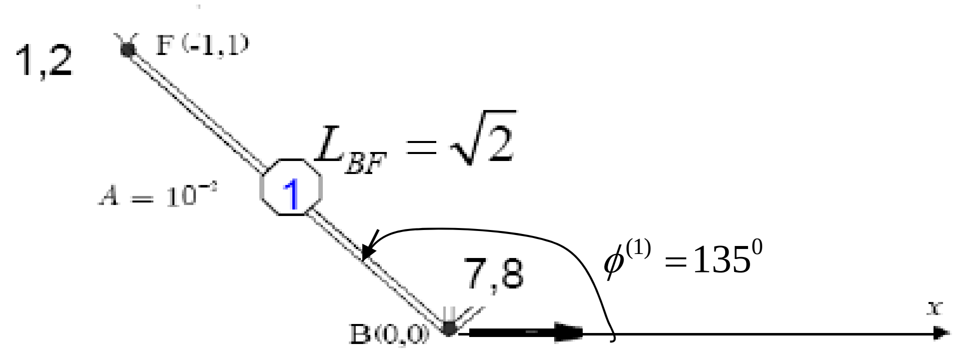


Note that point D is free to move in the *x* direction

$E = 10^7$Pa

- Construct the global stiffness matrix and load vector
- Partition the matrices and solve for the unknown displacements at point B, and displacement in x direction at point D.
- Find the stresses in the three bars
- Find the reactions at C, D and F

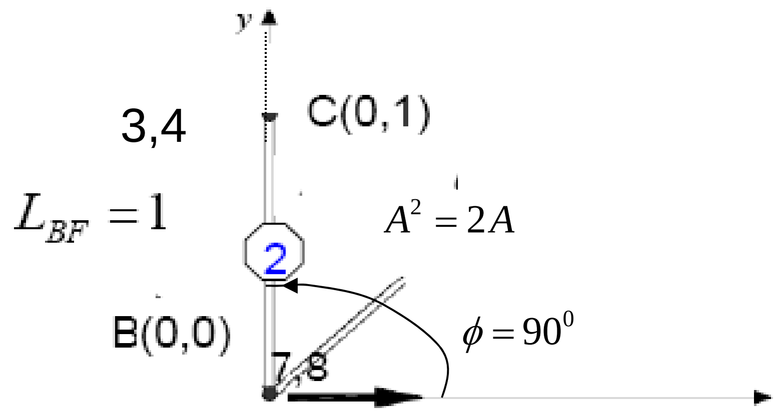
A truss example: Element 1



$$
\begin{array}{c}
\begin{array}{cccc} \quad\quad 1 & \quad\quad 2 & \quad\quad 7 & \quad\quad 8 \end{array} \\
[K^{(1)}] = \dfrac{EA}{\sqrt{2}} \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}
\end{array}
$$

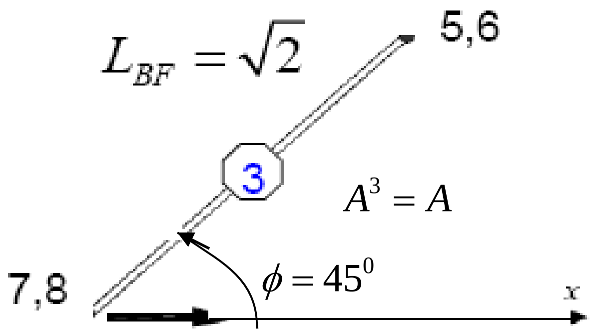
Note: Recall that you can number the corresponding global nodes in the sequence 7 8 1 2 without any changes in $[K^{(1)}]$.

A truss example: Element 2



$$
\begin{array}{ccccc}
 & 3 & 4 & 7 & 8 \\
[K^{(2)}]=\dfrac{EA}{\sqrt{2}} & \multicolumn{4}{c}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & -2\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & -2\sqrt{2} & 0 & 2\sqrt{2} \end{bmatrix}}
\end{array}
\begin{array}{c} 3 \\ 4 \\ 7 \\ 8 \end{array}
$$

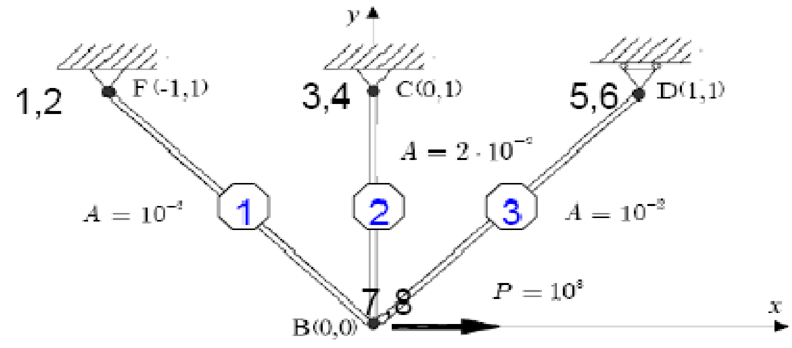
A truss example: Element 3



$$
\begin{array}{cc}
 & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\
[K^{(3)}] = \dfrac{EA}{\sqrt{2}} & \begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array}
\end{array}
$$

A truss example: Assembly (element 1)

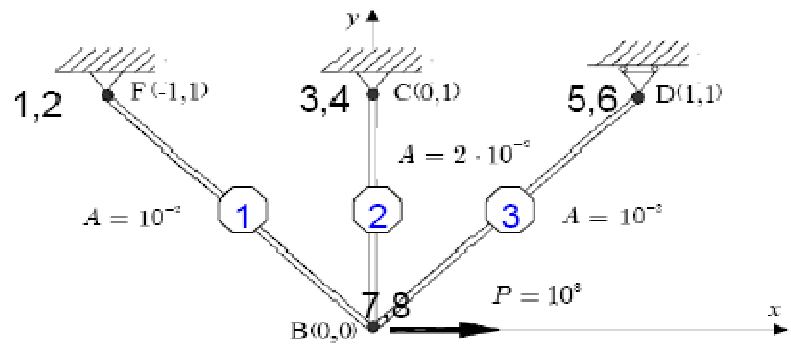
$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\
[K^{(1)}]=\dfrac{EA}{\sqrt{2}} & \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}
\end{array}
$$



$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
[K]=\dfrac{EA}{\sqrt{2}} & \begin{bmatrix}
1/2 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & -1/2 \\
-1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\
1/2 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & 1/2
\end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\end{array}
$$

A truss example: Assembly (element 2)

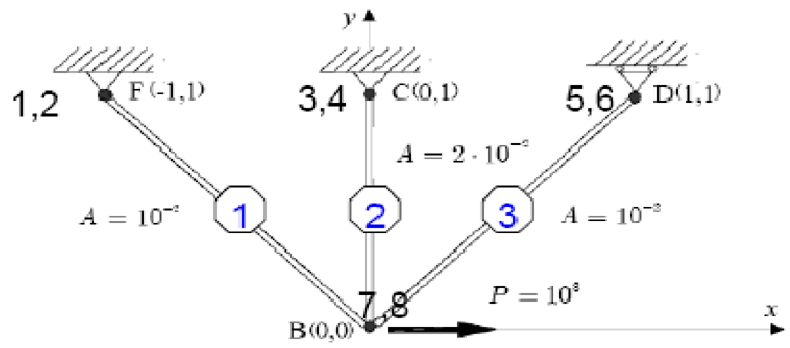
$$[K^{(2)}] = \frac{EA}{\sqrt{2}} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 & -2\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & -2\sqrt{2} & 0 & 2\sqrt{2} \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 7 \\ 8 \end{array}$$



$$[K] = \frac{EA}{\sqrt{2}} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & -2\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2+0 & -1/2+0 \\ 1/2 & -1/2 & 0 & -2\sqrt{2} & 0 & 0 & -1/2 & 1/2+2\sqrt{2} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

A truss example: Assembly (element 3)

$$
[K^{(3)}] = \frac{EA}{\sqrt{2}}
\begin{array}{c}
\begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
1/2 & 1/2 & -1/2 & -1/2 \\
1/2 & 1/2 & -1/2 & -1/2 \\
-1/2 & -1/2 & 1/2 & 1/2 \\
-1/2 & -1/2 & 1/2 & 1/2
\end{bmatrix}
\end{array}
\begin{array}{c} \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$



$$
[K] = \frac{EA}{\sqrt{2}}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
1/2 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & -1/2 \\
-1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & -2\sqrt{2} \\
0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\
0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\
-1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 & 1/2+0+1/2 & -1/2+0+1/2 \\
1/2 & -1/2 & 0 & -2\sqrt{2} & -1/2 & -1/2 & -1/2+1/2 & 1/2+2\sqrt{2}+1/2
\end{bmatrix}
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
$$

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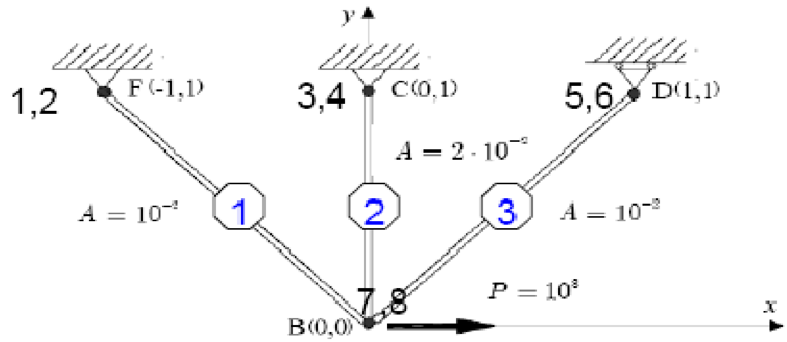
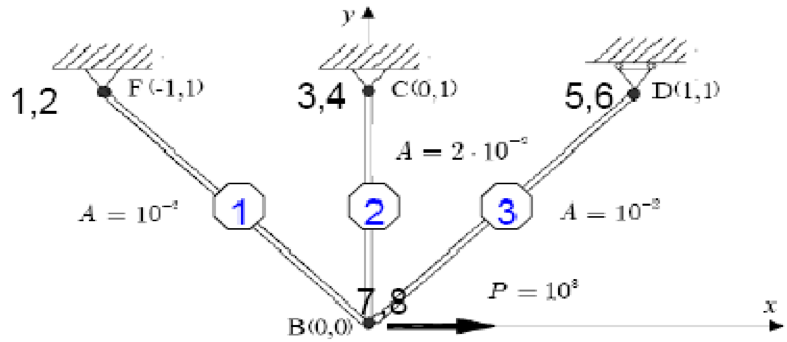
A truss example: Assembly



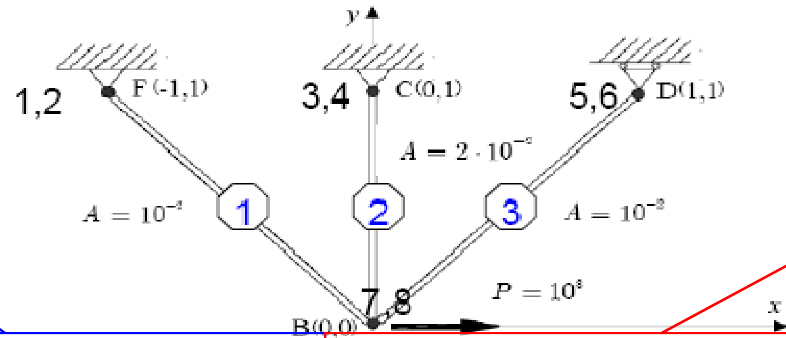
$$[K]=\frac{EA}{\sqrt{2}}\begin{array}{cc} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} & \\ \begin{bmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & -2\sqrt{2} \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 & 1 & 0 \\ 1/2 & -1/2 & 0 & -2\sqrt{2} & -1/2 & -1/2 & 0 & 1+2\sqrt{2} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array}$$

A truss example: Partition and BCs



$$\frac{EA}{\sqrt{2}}\begin{bmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & -2\sqrt{2} \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 & 1 & 0 \\ 1/2 & -1/2 & 0 & -2\sqrt{2} & -1/2 & -1/2 & 0 & 1+2\sqrt{2} \end{bmatrix}\begin{Bmatrix} d_1 = 0 \\ d_2 = 0 \\ d_3 = 0 \\ d_4 = 0 \\ d_5 \\ d_6 = 0 \\ d_7 \\ d_8 \end{Bmatrix}=\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ 0 \\ r_6 \\ 10^3\,Nt \\ 0 \end{Bmatrix}$$

A truss example: Partition and BCs

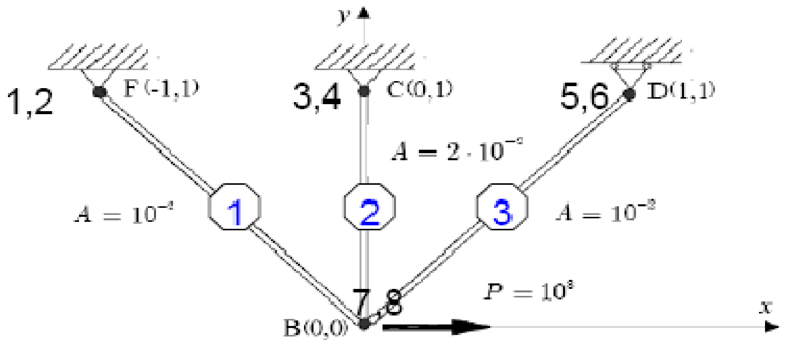


$[K_E]$ $\quad$ $[K_{EF}]$ $\quad$ $\bar{d}_E = 0$ $\quad$ $\bar{f}_E$

$$\frac{EA}{\sqrt{2}}\left[\begin{array}{ccccc|ccc} 1/2 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & -2\sqrt{2} \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ \hline 0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 & 1 & 0 \\ -1/2 & -1/2 & 0 & -2\sqrt{2} & -1/2 & -1/2 & 0 & 1+2\sqrt{2} \end{array}\right] \left\{\begin{array}{c} d_1 = 0 \\ d_2 = 0 \\ d_3 = 0 \\ d_4 = 0 \\ d_6 = 0 \\ \hline d_5 \\ d_7 \\ d_8 \end{array}\right\} = \left\{\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_6 \\ \hline 0 \\ 10^3\,Nt \\ 0 \end{array}\right\}$$

$[K^T_{EF}]$ $\quad$ $[K_F]$ $\quad$ d_F $\quad$ f_F

A truss example: Partition and BCs

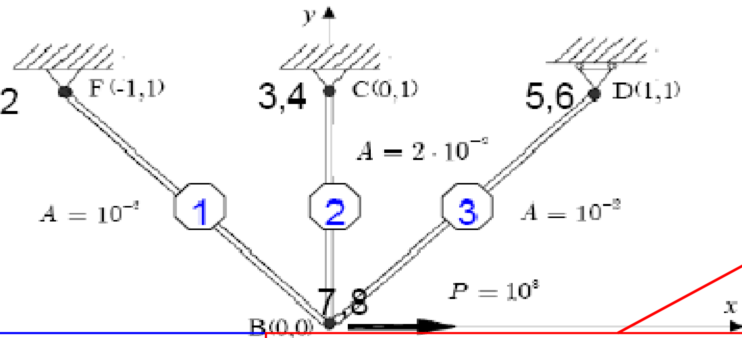


$$\underbrace{\frac{EA}{\sqrt{2}}\begin{bmatrix} 1/2 & -1/2 & -1/2 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1+2\sqrt{2} \end{bmatrix}}_{K_F}\underbrace{\begin{Bmatrix} d_5 \\ d_7 \\ d_8 \end{Bmatrix}}_{d_F} = \underbrace{\begin{Bmatrix} 0 \\ 10^3\, Nt \\ 0 \end{Bmatrix}}_{f_F}$$

$$\begin{Bmatrix} d_5 \\ d_7 \\ d_8 \end{Bmatrix} = \begin{Bmatrix} 0.038284m \\ 0.033284m \\ 0.005m \end{Bmatrix}$$

A truss example: Reaction calculation

$$\overline{f}_E = K_E \overline{d}_E + K_{EF} d_F$$

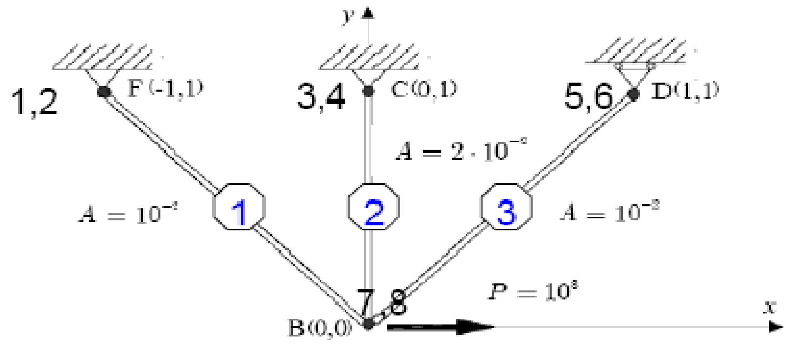


$[K_E]$ $[K_{EF}]$ $\overline{d}_E = 0$ $\overline{f}_E$

$$\frac{EA}{\sqrt{2}}\left[\begin{array}{ccccc|ccc}
1/2 & -1/2 & 0 & 0 & 0 & 0 & -1/2 & -1/2 \\
-1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\sqrt{2} & 0 & 0 & 0 & -2\sqrt{2} \\
0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\
\hline
0 & 0 & 0 & 0 & 1/2 & 1/2 & -1/2 & -1/2 \\
-1/2 & 1/2 & 0 & 0 & -1/2 & -1/2 & 1 & 0 \\
1/2 & -1/2 & 0 & -2\sqrt{2} & -1/2 & -1/2 & 0 & 1+2\sqrt{2}
\end{array}\right]
\left\{\begin{array}{c}
d_1 = 0 \\ d_2 = 0 \\ d_3 = 0 \\ d_4 = 0 \\ d_6 = 0 \\ \hline d_5 \\ d_7 \\ d_8
\end{array}\right\}
=
\left\{\begin{array}{c}
r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_6 \\ \hline 0 \\ 10^3\,Nt \\ 0
\end{array}\right\}$$

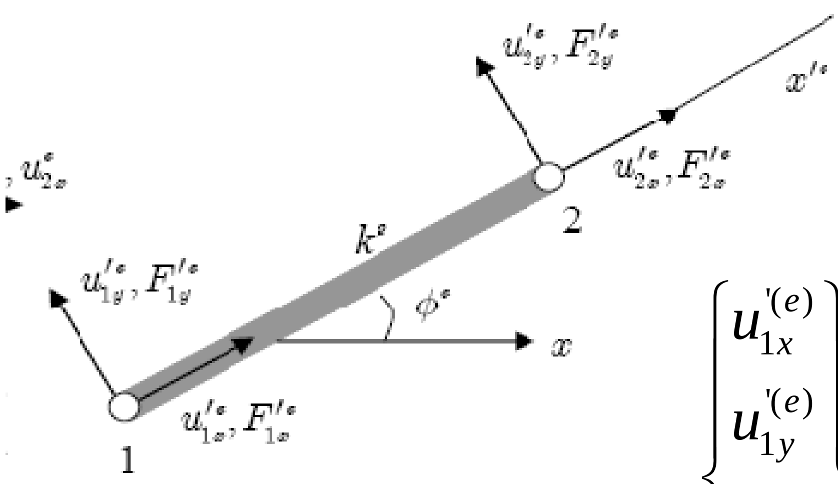
$[K_{EF}^T]$ $[K_F]$ d_F f_F

A truss example: Reaction calculation



$$\underbrace{\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_6 \end{Bmatrix}}_{f_E} = \underbrace{\frac{EA}{\sqrt{2}} \begin{bmatrix} 0 & -1/2 & 1/2 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & -2\sqrt{2} \\ 1/2 & -1/2 & -1/2 \end{bmatrix}}_{K_{EF}} \underbrace{\begin{Bmatrix} d_5 \\ d_7 \\ d_8 \end{Bmatrix}}_{d_F} \Rightarrow \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_6 \end{Bmatrix} \equiv \begin{Bmatrix} R_{Fx} \\ R_{Fy} \\ R_{Cx} \\ R_{Cy} \\ R_{Dy} \end{Bmatrix} = \begin{Bmatrix} -1000Nt \\ 1000Nt \\ 0 \\ -1000Nt \\ 0 \end{Bmatrix}$$

A truss example: Compute the stresses



$$\varepsilon^e = \frac{u'^e_{2x} - u'^e_{1x}}{L^e} \Rightarrow \sigma^e = E^e \frac{u'^e_{2x} - u'^e_{1x}}{L^e}$$

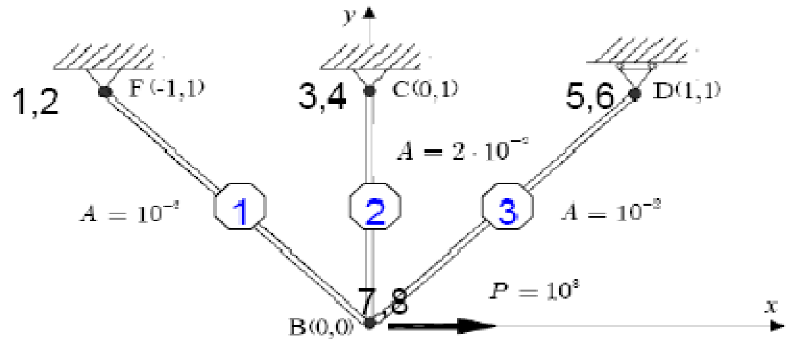
However:

$$\begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{1y} \\ u'^{(e)}_{2x} \\ u'^{(e)}_{2y} \end{Bmatrix} = \begin{bmatrix} \cos\phi^e & \sin\phi^e & 0 & 0 \\ -\sin\phi^e & \cos\phi^e & 0 & 0 \\ 0 & 0 & \cos\phi^e & \sin\phi^e \\ 0 & 0 & -\sin\phi^e & \cos\phi^e \end{bmatrix} \underbrace{\begin{Bmatrix} u^{(e)}_{1x} \\ u^{(e)}_{1y} \\ u^{(e)}_{2x} \\ u^{(e)}_{2y} \end{Bmatrix}}_{\{d^e\}}$$

Combining the 2 Eqs gives:

$$\sigma^e = \frac{E^e}{L^e} \begin{bmatrix} -\cos\phi^e & -\sin\phi^e & \cos\phi^e & \sin\phi^e \end{bmatrix} \{d^e\}$$

A truss example: Compute the stresses



$$\sigma^e = \frac{E^e}{L^e}\begin{bmatrix} -\cos\phi^e & -\sin\phi^e & \cos\phi^e & \sin\phi^e \end{bmatrix}\{d^e\}$$

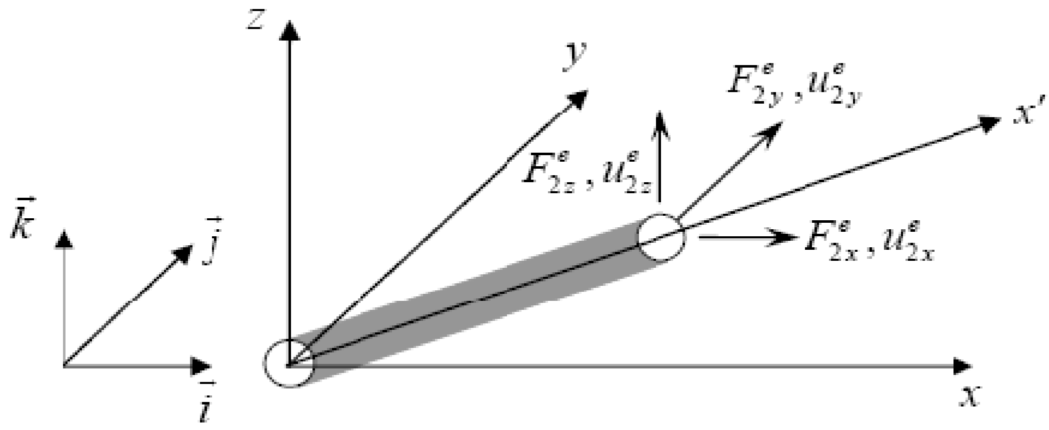
- Applying this to each element, we have:

$$\sigma^{(1)} = \frac{E}{\sqrt{2}}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.038284m \\ 0.005m \end{Bmatrix} = -141.421kPa$$

$$\sigma^{(2)} = E\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.033284m \\ 0.005m \end{Bmatrix} = 50kPa$$

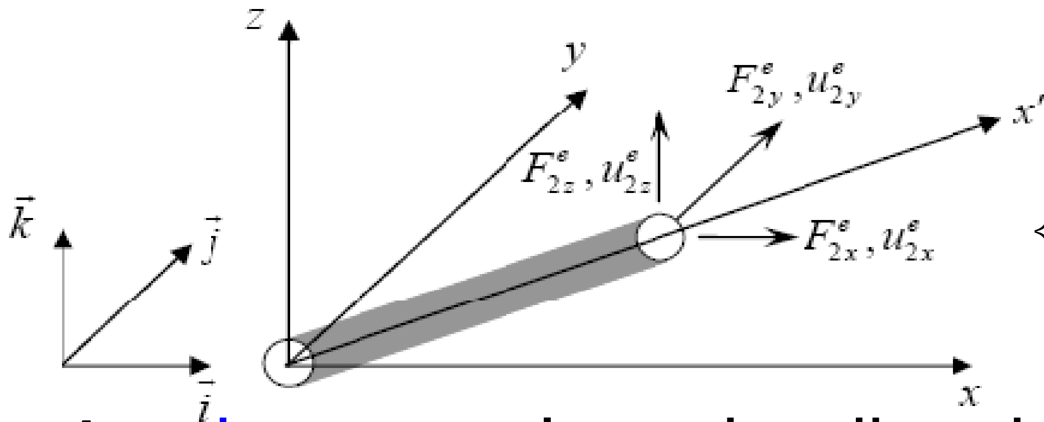
$$\sigma^{(3)} = \frac{E}{\sqrt{2}}\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{Bmatrix} 0.038284m \\ 0 \\ 0.033284m \\ 0.005m \end{Bmatrix} = 0kPa$$

Three-dimensional (space) truss structures



- The local stiffness eqs are exactly as before, i.e. $\begin{Bmatrix} F'^{(e)}_{1x} \\ F'^{(e)}_{2x} \end{Bmatrix} = \begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{2x} \end{Bmatrix}, \quad k^{(e)} = A^e E^e / L^e$
- However, we now have to include displacements and forces in the *x*, *y* and *z* axes.

Three-dimensional (space) truss structures



$$\begin{Bmatrix} F_{1x}^{'(e)} \\ F_{2x}^{'(e)} \end{Bmatrix} = \begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u_{1x}^{'(e)} \\ u_{2x}^{'(e)} \end{Bmatrix}$$

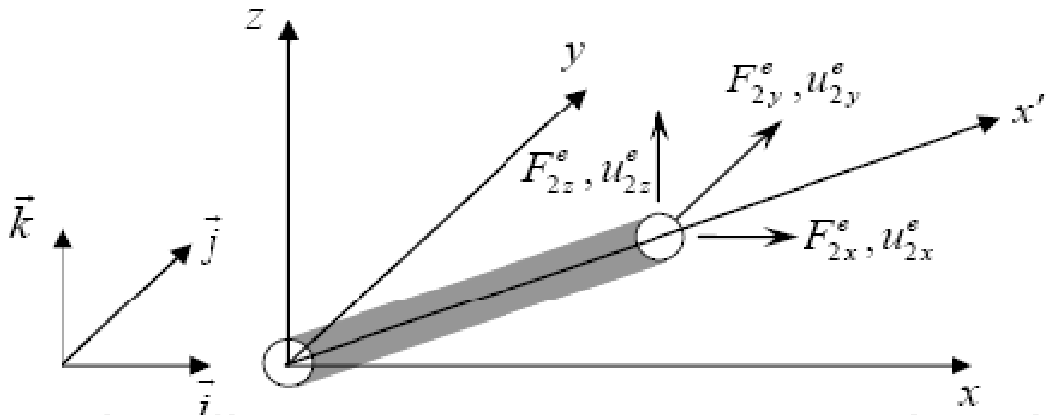
- A unit vector along the direction x' of a 3D truss element has the components (direction cosines of the axes between x' and x,y,z, respectively):

$$l^e_s = \frac{x^e_2 - x^e_1}{L^e}, \quad m^e_s = \frac{y^e_2 - y^e_1}{L^e}, \quad n^e_s = \frac{z^e_2 - z^e_1}{L^e}$$

$$L^e = \sqrt{(x^e_2 - x^e_1)^2 + (y^e_2 - y^e_1)^2 + (z^e_2 - z^e_1)^2}$$

where (x^e_1, y^e_1, z^e_1) *and* (x^e_2, y^e_2, z^e_2) are the nodal coordinates in the (x,y,z) system.

Three-dimensional (space) truss structures



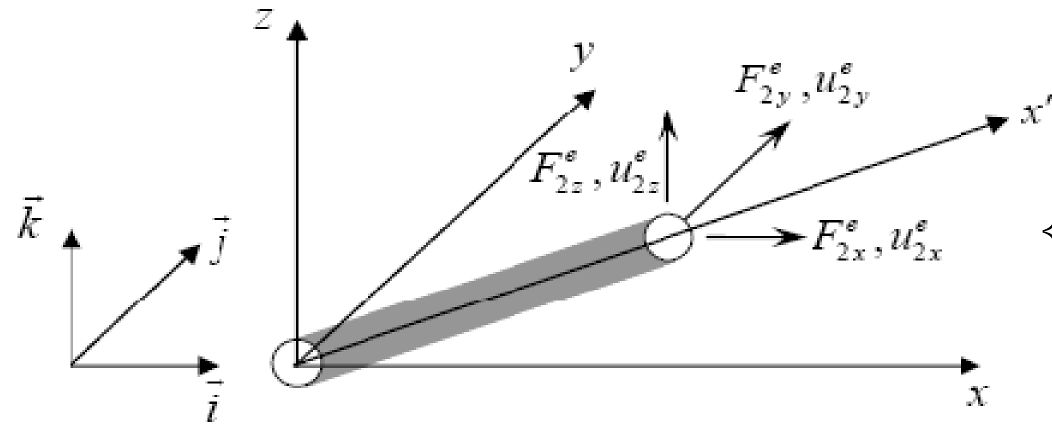
$$\begin{Bmatrix} F'^{(e)}_{1x} \\ F'^{(e)}_{2x} \end{Bmatrix} = \begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{2x} \end{Bmatrix}$$

- The displacement transformation then takes the form:

$$\begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{2x} \end{Bmatrix} = \underbrace{\begin{bmatrix} l^e_s & m^e_s & n^e_s & 0 & 0 & 0 \\ 0 & 0 & 0 & l^e_s & m^e_s & n^e_s \end{bmatrix}}_{[T^e]} \underbrace{\begin{Bmatrix} u^{(e)}_{1x} \\ u^{(e)}_{1y} \\ u^{(e)}_{1z} \\ u^{(e)}_{2x} \\ u^{(e)}_{2y} \\ u^{(e)}_{2z} \end{Bmatrix}}_{\{d^e\}} \equiv [T^e]\{d^e\}$$

- Similar transformation is applied for the forces: $\begin{Bmatrix} F'^{(e)}_{1x} \\ F'^{(e)}_{2x} \end{Bmatrix} = [T^e]\{F^e\}$

Three-dimensional (space) truss structures



$$\begin{Bmatrix} F'^{(e)}_{1x} \\ F'^{(e)}_{2x} \end{Bmatrix} = \begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{2x} \end{Bmatrix}$$

- Similarly to the derivation for 2D trusses, the stiffness matrix in global coordinates is then:

$$\underbrace{[K]}_{6x6} \equiv \underbrace{[T^e]^T}_{6x2} \quad \underbrace{\{K^e\}}_{2x2} \quad \underbrace{[T^e]}_{2x6}$$

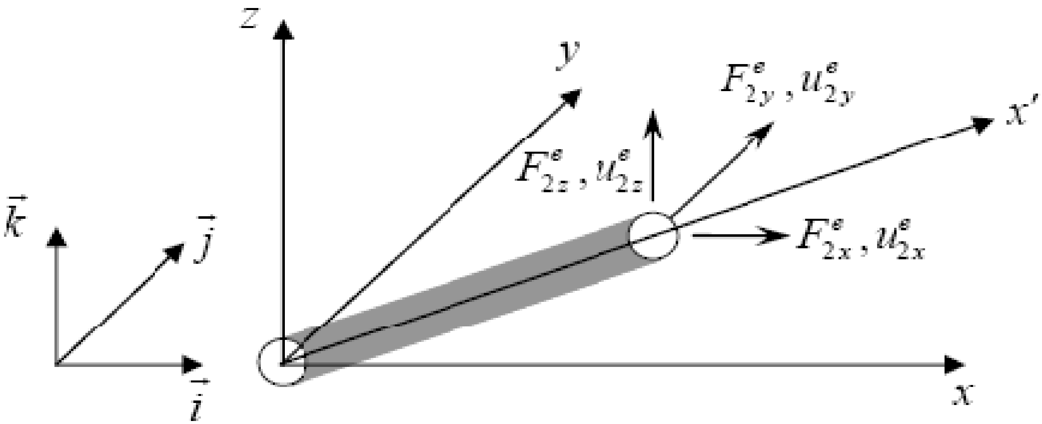
Stiffness of a space element

$$[K^{(e)}] = \frac{E^e A^e}{L^e} \begin{bmatrix} l_s^{e2} & m_s^e l_s^e & n_s^e l_s^e & -l_s^{e2} & -m_s^e l_s^e & -n_s^e l_s^e \\ m_s^e l_s^e & m_s^{e2} & m_s^e n_s^e & -m_s^e l_s^e & -m_s^{e2} & -m_s^e n_s^e \\ n_s^e l_s^e & m_s^e n_s^e & n_s^{e2} & -n_s^e l_s^e & -m_s^e n_s^e & -n_s^{e2} \\ -l_s^{e2} & -m_s^e l_s^e & -n_s^e l_s^e & l_s^{e2} & m_s^e l_s^e & n_s^e l_s^e \\ -m_s^e l_s^e & -m_s^{e2} & -m_s^e n_s^{e2} & m_s^e l_s^e & m_s^{e2} & m_s^e n_s^e \\ -n_s^e l_s^e & -m_s^e n_s^e & -n_s^{e2} & n_s^e l_s^e & m_s^e n_s^e & n_s^{e2} \end{bmatrix}$$

$$where: \quad l_s^e = \frac{x_2^e - x_1^e}{L^e}, \quad m_s^e = \frac{y_2^e - y_1^e}{L^e}, \quad n_s^e = \frac{z_2^e - z_1^e}{L^e}$$

$$L^e = \sqrt{(x_2^e - x_1^e)^2 + (y_2^e - y_1^e)^2 + (z_2^e - z_1^e)^2}$$

Computing the stresses in a space truss element



$$\varepsilon^e = \frac{u'^e_{2x} - u'^e_{1x}}{L^e} \Rightarrow$$

$$\sigma^e = E^e \frac{u'^e_{2x} - u'^e_{1x}}{L^e} = \frac{E^e}{L^e}\left(u'^e_{2x} - u'^e_{1x}\right)$$

- However:

$$\begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{2x} \end{Bmatrix} = \begin{bmatrix} l^e_s & m^e_s & n^e_s & 0 & 0 & 0 \\ 0 & 0 & 0 & l^e_s & m^e_s & n^e_s \end{bmatrix} \begin{Bmatrix} u^{(e)}_{1x} \\ u^{(e)}_{1y} \\ u^{(e)}_{1z} \\ u^{(e)}_{2x} \\ u^{(e)}_{2y} \\ u^{(e)}_{2z} \end{Bmatrix} \equiv [T^e]\{d^e\}$$

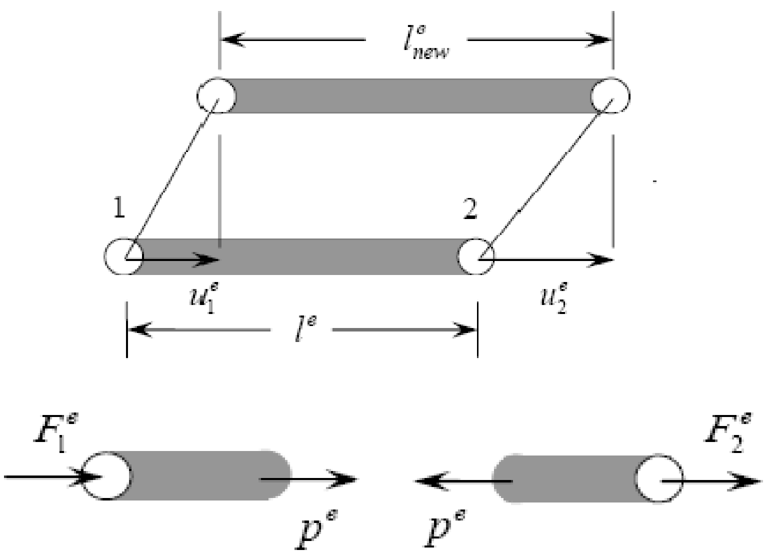
- Combining the 2 Eqs gives:

$$\sigma^e = \frac{E^e}{L^e}\begin{bmatrix} -l^e_s & -m^e_s & -n^e_s & l^e_s & m^e_s & n^e_s \end{bmatrix}\{d^e\}$$

Accounting for thermal effects in truss analysis

- Consider a truss structure that is heated. We need to account for thermal expansion effects. Note: $\underbrace{\varepsilon^e}_{total\ strain} = \varepsilon^e_{elastic} + \varepsilon^e_{thermal}$
- Hooke's law is now modified as follows (using the x' coordinate system):

$$\sigma^e = E^e \varepsilon^e_{elastic} = E^e(\underbrace{\varepsilon^e}_{total\ strain} - \varepsilon^e_{thermal}) = E^e\left(\frac{u_2^e - u_1^e}{L^e} - \alpha^e \Delta T^e\right)$$



$$F_2^{(e)} = -F_1^{(e)} = p^e = A^e \sigma^e = A^e E^e\left(\frac{u_2^e - u_1^e}{L^e} - \underbrace{\alpha^e \Delta T^e}_{\varepsilon_0^e}\right) =$$

$$= k^e(u_2^e - u_1^e) - A^e E^e \varepsilon_0^e, \; k^e = \frac{A^e E^e}{L^e}$$

$$\begin{Bmatrix} F_1^{(e)} \\ F_2^{(e)} \end{Bmatrix} + \underbrace{A^e E^e \varepsilon_0^e \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}}_{Thermal\ nodal\ vector} = \begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix}$$

Element equations with thermal effects

- Expanding these equations to include nodal displacements in the *y*' axis gives:

$$\underbrace{\begin{Bmatrix} F'^{(e)}_{1x} \\ F'^{(e)}_{1y} \\ F'^{(e)}_{2x} \\ F'^{(e)}_{2y} \end{Bmatrix}}_{\{F'^{(e)}\}} + \underbrace{A^e E^e \varepsilon_0^e \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix}}_{\{F'^{e}_{thermal}\}} = \underbrace{k^{(e)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{[K'^{(e)}]} \underbrace{\begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{1y} \\ u'^{(e)}_{2x} \\ u'^{(e)}_{2y} \end{Bmatrix}}_{\{d'^{(e)}\}}$$

- We need to transform this to *x* and *y* displacements (our degrees of freedom for this element)

$$\{d'^e\} = [T^e]\{d^e\} \qquad \{F'^e\} = [T^e]\{F^e\}$$

Element equations with thermal effects

$$\underbrace{\begin{Bmatrix} F_{1x}^{'(e)} \\ F_{1y}^{'(e)} \\ F_{2x}^{'(e)} \\ F_{2y}^{'(e)} \end{Bmatrix}}_{\{F^{'(e)}\}} + \underbrace{A^e E^e \varepsilon_0^e \begin{Bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{Bmatrix}}_{\{F^{'e}_{thermal}\}} = \underbrace{k^{(e)} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{[K^{'(e)}]} \underbrace{\begin{Bmatrix} u_{1x}^{'(e)} \\ u_{1y}^{'(e)} \\ u_{2x}^{'(e)} \\ u_{2y}^{'(e)} \end{Bmatrix}}_{\{d^{'(e)}\}}$$

$$\{d^{'e}\} = [T^e]\{d^e\}$$

$$\{F^{'e}\} = [T^e]\{F^e\}$$

- We can transform these element equations as follows:

$$[T^e]\{F^e\} + \{F^{'e}_{thermal}\} = [K^{'(e)}][T^e]\{d^e\}$$

- From these equations, we conclude that:

$$\{F^e\} + \underbrace{[T^e]^T \{F^{'e}_{thermal}\}}_{\{F^e_{thermal}\}} = \underbrace{[T^e]^T [K^{'(e)}][T^e]}_{[K^{(e)}]}\{d^e\}$$

Element equations with thermal effects

$$\{F^e\} + \underbrace{[T^e]^T \{F'^e_{thermal}\}}_{\{F^e_{thermal}\}} = \underbrace{[T^e]^T [K'^{(e)}][T^e]}_{[K^{(e)}]}\{d^e\}$$

Use: $$[T^{(e)}] = \begin{bmatrix} \cos\phi^e & \sin\phi^e & 0 & 0 \\ -\sin\phi^e & \cos\phi^e & 0 & 0 \\ 0 & 0 & \cos\phi^e & \sin\phi^e \\ 0 & 0 & -\sin\phi^e & \cos\phi^e \end{bmatrix}$$ (for 2D trusses)

Finally we obtain:

$$\underbrace{\begin{Bmatrix} F_{1x}^{(e)} \\ F_{1y}^{(e)} \\ F_{2x}^{(e)} \\ F_{2y}^{(e)} \end{Bmatrix}}_{\{F^{(e)}\}} + \underbrace{A^e E^e \varepsilon_0^e \begin{Bmatrix} -\cos\phi^e \\ -\sin\phi^e \\ \cos\phi^e \\ \sin\phi^e \end{Bmatrix}}_{\{F^e_{thermal}\}} = \underbrace{k^{(e)} \begin{bmatrix} \cos^2\phi^e & \sin\phi^e\cos\phi^e & -\cos^2\phi^e & -\sin\phi^e\cos\phi^e \\ \sin\phi^e\cos\phi^e & \sin^2\phi^e & -\sin\phi^e\cos\phi^e & -\sin^2\phi^e \\ -\cos^2\phi^e & -\sin\phi^e\cos\phi^e & \cos^2\phi^e & \sin\phi^e\cos\phi^e \\ -\sin\phi^e\cos\phi^e & -\sin^2\phi^e & \sin\phi^e\cos\phi^e & \sin^2\phi^e \end{bmatrix}}_{[K^{(e)}]} \underbrace{\begin{Bmatrix} u_{1x}^{(e)} \\ u_{1y}^{(e)} \\ u_{2x}^{(e)} \\ u_{2y}^{(e)} \end{Bmatrix}}_{d^{(e)}}$$

Element equations with thermal effects

- What do you need to do to account for thermal effects in truss analysis?

- For each truss element that is heated, simply add to the element force, the following extra term

$$\underbrace{A^e E^e \varepsilon_0^e \begin{Bmatrix} -\cos\phi^e \\ -\sin\phi^e \\ \cos\phi^e \\ \sin\phi^e \end{Bmatrix}}_{\{F^e_{thermal}\}}, \; where \; \varepsilon_0^e = \alpha^e \Delta T^e$$

- You will need to define at which truss elements thermal effects take place and for each of them read the values α^e and ΔT^e.

Principle of minimum potential energy

- An alternative equivalent approach to solving many structural problems is the principle of minimum potential energy.

From all the possible compatible displacements of a structure, the one that minimizes the total potential energy is the exact solution.

Potential energy for given displacements	=	Strain energy for these given displacements	-	Work done by external loads on these given displacements

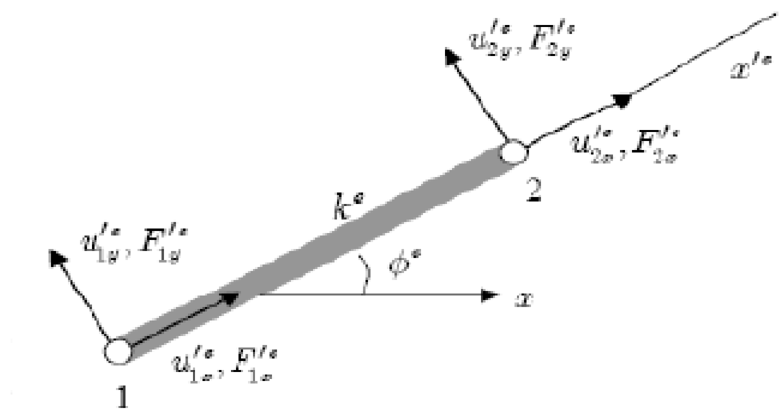
Principle of minimum potential energy

- Let us see this principle applied to the truss problems discussed earlier.

Assembly process

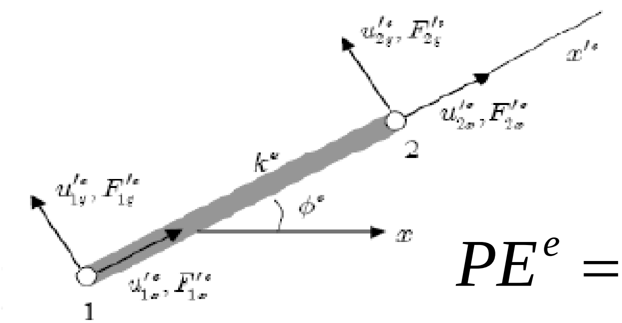
$$\min_{\{d\}} \sum_{e} PE^{e}, \; PE^{e} = U^{e} - W^{e}$$

$$PE^{e} = \int_{\Omega^{e}} \underbrace{\frac{1}{2}\sigma^{e}\varepsilon^{e}}_{\substack{\text{Elastic strain energy density} \\ (\text{work/volume})}} dV^{e} - \underbrace{\left(F_{1x}^{'(e)}u_{1x}^{'(e)} + F_{2x}^{'(e)}u_{2x}^{'(e)}\right)}_{\text{External Work}}$$



Principle of minimum potential energy

- Lets apply this principle to one truss element. We need to minimize with respect to the nodal displacements (local coordinates) $u_{1x}^{'(e)}, u_{2x}^{'(e)}$. Recall that:

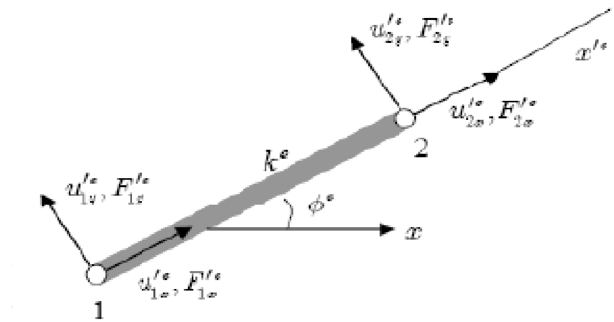


$$\varepsilon^e = \frac{u_{2x}^{'e} - u_{1x}^{'e}}{L^e}, \sigma^e = E^e \varepsilon^e$$

$$PE^e = \int_{\Omega^e} \frac{1}{2} E^e \varepsilon^{e2} dV^e - F_{1x}^{'(e)} u_{1x}^{'(e)} - F_{2x}^{'(e)} u_{2x}^{'(e)} =$$

$$\int_{\Omega^e} \frac{1}{2} E^e \left(\frac{u_{2x}^{'e} - u_{1x}^{'e}}{L^e}\right)^2 \underbrace{dV^e}_{A^e dx'} - F_{1x}^{'(e)} u_{1x}^{'(e)} - F_{2x}^{'(e)} u_{2x}^{'(e)}$$

Principle of minimum potential energy



$$\min_{u'^{e}_{1x},\,u'^{e}_{2x}} PE^e = \min_{u'^{e}_{1x},\,u'^{e}_{2x}} E^e A^e L^e \frac{1}{2}\left(\frac{u'^{e}_{2x} - u'^{e}_{1x}}{L^e}\right)^2 - F'^{(e)}_{1x} u'^{(e)}_{1x} - F'^{(e)}_{2x} u'^{(e)}_{2x}$$

- Take partial derivatives of PE^e wrt u'^{e}_{1x}, u'^{e}_{2x} :

$$\frac{\partial PE^e}{\partial u'^{e}_{1x}} = 0 \Rightarrow \frac{E^e A^e}{L^e}\left(u'^{e}_{1x} - u'^{e}_{2x}\right) - F'^{(e)}_{1x} = 0$$

$$\frac{\partial PE^e}{\partial u'^{e}_{2x}} = 0 \Rightarrow \frac{E^e A^e}{L^e}\left(u'^{e}_{2x} - u'^{e}_{1x}\right) - F'^{(e)}_{2x} = 0$$

$$\Rightarrow \begin{Bmatrix} F'^{(e)}_{1x} \\ F'^{(e)}_{2x} \end{Bmatrix} = \begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u'^{(e)}_{1x} \\ u'^{(e)}_{2x} \end{Bmatrix}$$

These are the same Eqs as those obtained with the direct method!

Principle of minimum potential energy

- In general (not just for mechanics problems!), the principle of minimum potential energy takes the following form:

$$\min_{\{d\}} \sum_e PE^e = \min_{\{d\}} \left(\sum_e \left(\frac{1}{2} \{d^{(e)}\}^T [K^{(e)}] \{d^{(e)}\} \right) - \{d\}^T \{F\} \right)$$

  or after assembly:

$$\min_{\{d\}} PE = \min_{\{d\}} \left(\frac{1}{2} \{d\}^T [K] \{d\} - \{d\}^T \{F\} \right)$$

- Note that the mimimization gives us the familiar solution: $[K]\{d\} = \{F\}$

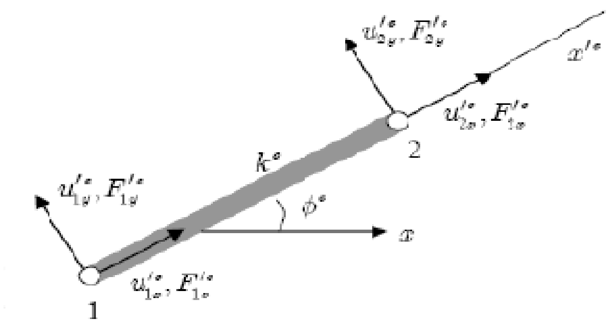
Principle of minimum potential energy

- We will not use this method to repeat the truss calculations.
- However, it will be our starting point for computing the stiffness of beam elements (lecture 4).
- The method of minimum potential energy can be applied to many problems not related to mechanics – however there are many problems where this technique is not applicable.
- After discussing beam bending problems, we will need to look for more powerful (`unfortunately' also more mathematical) methods (weak (Galerkin) formulations).

Revisiting the 2-node truss element

- Up to now we used the direct method to express the nodal loads vs. nodal displacements for the 2-node truss element.
- Let us linearly interpolate the displacement $u_x^{'(e)}$ of any point *x'* in the element in terms of the nodal displacements:

$$u_x^{'(e)} = (1-\frac{x'}{L^e})u_{1x}^{'(e)} + \frac{x'}{L^e}u_{2x}^{'(e)} = \underbrace{[\underbrace{1-\frac{x'}{L^e}}_{N_1^{(e)}}, \underbrace{\frac{x'}{L^e}}_{N_2^{(e)}}]}_{\text{element basis functions}} \underbrace{\begin{Bmatrix} u_{1x}^{'(e)} \\ u_{2x}^{'(e)} \end{Bmatrix}}_{\text{Nodal displacements}} = \underbrace{[N^{(e)}]}_{\text{basis functions matrix}} \{d^{(e)}\}$$



- The strain in this 2-node element $\varepsilon^e = \frac{du_x^{'e}}{dx'}$ can now be computed as follows:

$$u_x^{'(e)} = [N^{(e)}]\{d^{(e)}\} \Rightarrow \varepsilon^e = \frac{du_x^{'e}}{dx'} = \frac{d[N^{(e)}]\{d^{(e)}\}}{dx'} = [\frac{dN^{(e)}}{dx'}]\{d^{(e)}\} \equiv [B^{(e)}]\{d^{(e)}\}$$

$$\underbrace{[B^{(e)}]}_{\text{Gradient of basis functions matrix}} = [\frac{dN_1^{(e)}}{dx'}, \frac{dN_2^{(e)}}{dx'}] = [-\frac{1}{L^e}, \frac{1}{L^e}] \Rightarrow \varepsilon^e = [-\frac{1}{L^e}, \frac{1}{L^e}]\begin{Bmatrix} u_{1x}^{'(e)} \\ u_{2x}^{'(e)} \end{Bmatrix} = \frac{u_{2x}^{'(e)} - u_{1x}^{'(e)}}{L^e}$$

- This is exactly the same approximation we used before (constant strain element)

Revisiting the 2-node truss element

- Let us use these interpolation formulas $u^{'(e)}=[N^{(e)}]\{d^{(e)}\},\varepsilon^e=[B^{(e)}]\{d^{(e)}\}$ to express the potential energy in a format that will become very familiar as we proceed forward in this course.

$$PE^e=\int_{\Omega^e}\frac{1}{2}E^e\varepsilon^{e2}dV^e-F_{1x}^{'(e)}u_{1x}^{'(e)}-F_{2x}^{'(e)}u_{2x}^{'(e)}=$$

$$\int_{\Omega^e}\frac{1}{2}\underbrace{\left\{d^{(e)}\right\}^T[B^e]^T}_{\varepsilon^{eT}}E^e\underbrace{[B^e]\left\{d^{(e)}\right\}}_{\varepsilon^e}\underbrace{dV^e}_{A^e dx'}-F_{1x}^{'(e)}u_{1x}^{'(e)}-F_{2x}^{'(e)}u_{2x}^{'(e)}=$$

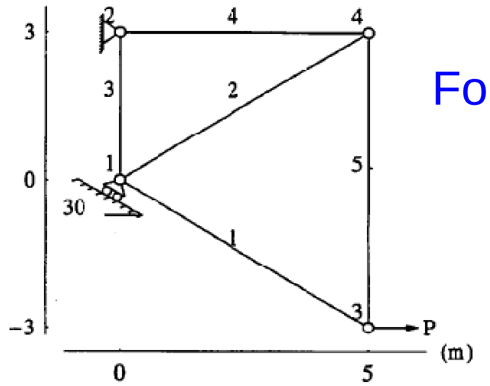
$$=\frac{1}{2}\left\{d^{(e)}\right\}^T\underbrace{\left[\int_{\Omega^e}[B^e]^T E^e[B^e]dV^e\right]}_{\substack{[K^e]\\ \text{Element stiffness}\\ \text{matrix}}}\left\{d^{(e)}\right\}-F_{1x}^{'(e)}u_{1x}^{'(e)}-F_{2x}^{'(e)}u_{2x}^{'(e)}$$

- For now these calculations are identical to our earlier approach! Indeed:

$$[K^e]=\int_{\Omega^e}[B^e]^T E^e[B^e]dV^e=\int_0^{L^e}[B^e]^T E^e[B^e]A^e dx'=\int_0^{L^e}\begin{bmatrix}-\frac{1}{L^e}\\ \frac{1}{L^e}\end{bmatrix}E^e[-\frac{1}{L^e},\frac{1}{L^e}]A^e dx'=\underbrace{\frac{A^e E^e}{L^e}}_{k^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Truss analysis with displacement constraints

- Up to this point, we imposed essential boundary conditions in terms of prescribed *x*- or *y*- nodal displacements. How about if the support is inclined as in the figure below:

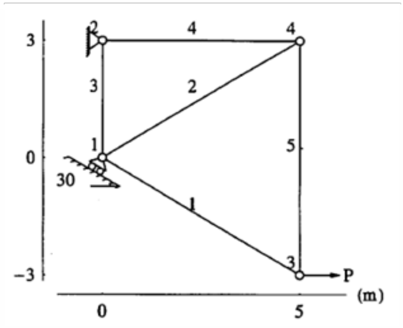


For this problem, the constraint is that there is no normal displacement at the support 1

- Here, we don't know the displacements at node 1 but we know the relation between u_{1x} and u_{1y}. In general we write these constrains on our nodal degrees of freedom as: $Cd=q$.

Truss analysis with displacement constraints

- Note that at node 1 we don't have essential boundary conditions – we have a displacement constraint.



- To solve this problem we use the principle of minimum potential energy with the constraint *Cd*=*q*:

Find **d** and λ such that

$$\min_{\{d\}} L = \underbrace{\frac{1}{2}\{d\}^T[K]\{d\} - \{d\}^T\{F\}}_{\substack{Potential\ energy \\ of\ unconstrained \\ system}} + \underbrace{\{\lambda\}}_{\substack{Lagrange \\ multiplier \\ enforcing \\ the\ constraint}}{}^T \underbrace{([C]\{d\} - \{q\})}_{Constraint}$$

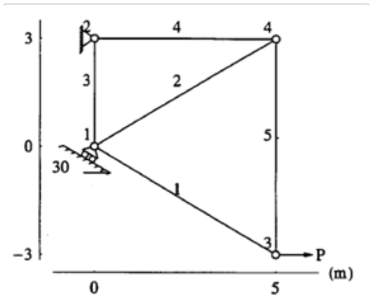
- Here, we enforce the constraint using Lagrange multipliers.

Truss analysis with displacement constraints

Find **d** and λ such that

$$\min_{\{d\}} L = \underbrace{\frac{1}{2}\{d\}^T[K]\{d\} - \{d\}^T\{F\}}_{\substack{\text{Potential energy}\\ \text{of unconstrained}\\ \text{system}}} + \underbrace{\{\lambda\}}_{\substack{\text{Lagrange}\\ \text{multiplier}\\ \text{enforcing}\\ \text{the constraint}}}{}^T \underbrace{([C]\{d\} - \{q\})}_{\text{Constraint}}$$

- λ is the Lagrange multiplier that enforces the constraint – it is nothing else but the reaction force at node 1 (normal to the support!)



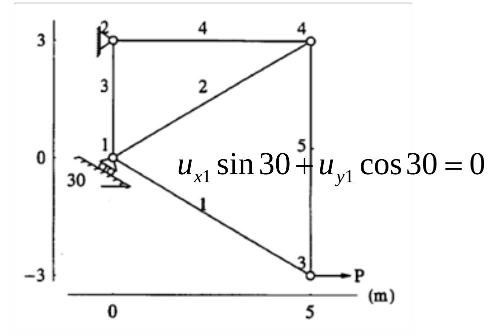
- Minimization is now performed with respect to both d and λ.

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix}\begin{pmatrix} d \\ \lambda \end{pmatrix} = \begin{pmatrix} F \\ q \end{pmatrix} \Rightarrow$$

Apply essential boundary conditions and then solve for $\{d_F\}$ and λ (here, reaction force at node 1)

Displacement constraints: Implementation

- How do you implement this in the MatLab libraries of HW1?



- Introduce the constraints in the InputData.m and then modify the stiffness and load vectors in the NodalSoln.m.
- Apply the essential boundary conditions first before you augment the reduced global equations (K_f) with the Lagrange multiplier.

InputData.m

```
                                        % Read information for constraints
C = zeros(1,neq-length(debc));          %The dimension of C is neq minus the
                                        % prescribed DOF via essential BCs
                                        % Here there is only one constraint
C(1) = sin(pi/6); C(2) = cos(pi/6);
q = 0;
```

Displacement constraints: NodalSoln.m

```
function [d, rf, lambda] = NodalSoln(K, R, debc, ebcVals, C, q)

% K=global stiffness, R=global force, debc=degrees of freedom with specified values, ebcVals=specified displacements

dof = length(R);              % Extract the total degreess-of-freedom

df = setdiff(1:dof, debc);    % Sets the difference between two sets of indices, i.e. the global degrees of freedom minus the
                              % degrees of freedom with prescribed essential boundary conditions

Kf = K(df, df);               % Remove eqs. corresponding to prescribed displacements

Rf = R(df) - K(df, debc)*ebcVals; % Modify the remaining load vector to account for the essential boundary conditions

[m n] = size(C);              % Extract number of constraints

Kf = [Kf  C';                 % Augment global equations with the Lagrange multipliers
      C  zeros(m)];

Rf = [Rf;q];                  % Augment load vector

dfVals = Kf\Rf;               % Solve the linear system of equations. Here for simplicity, we use Gauss elimination.

d = zeros(dof,1);             % Restore the solution vector (i.e. include back the nodes with prescribed displacements).
d(debc) = ebcVals;            % Use the originally established ordering of the nodes.
d(df) = dfVals(1:(length(dfVals)-m));

rf = K(debc,:)*d - R(debc);   % Calculate the reaction vector at nodes with prescribed displacements

lambda = dfVals((length(dfVals)-m+1):length(dfVals)); % Calculate Langrange multipliers (reactions at nodes with constraints)
```

$$\begin{pmatrix} K & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} d \\ \lambda \end{pmatrix} = \begin{pmatrix} F \\ q \end{pmatrix}$$