MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

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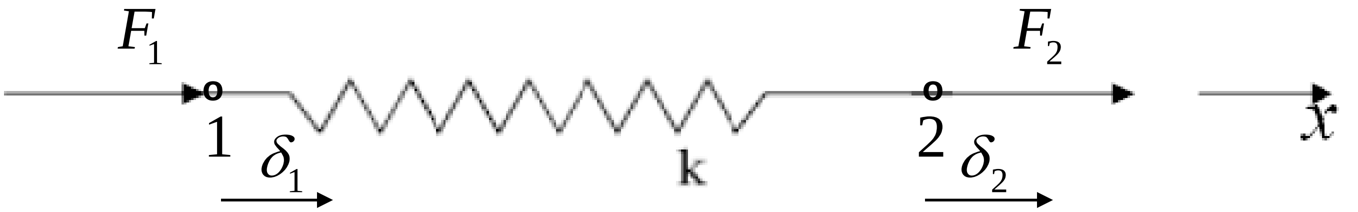
Intro to FEM: The direct approach

- Consider a linear spring with `nodes' 1 and 2:



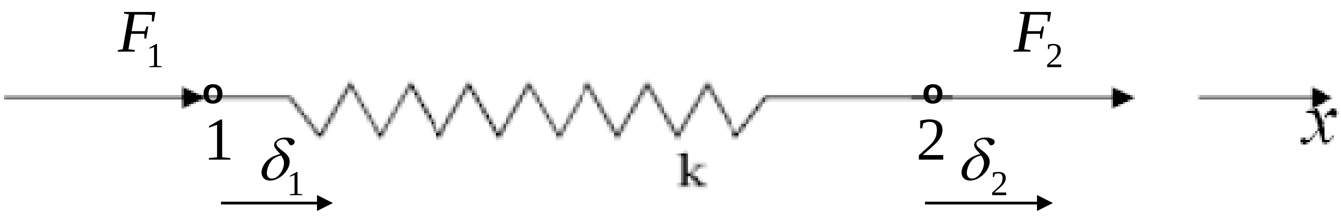
- The force on the spring at node 1 is: $F_1 = k(\delta_1 - \delta_2)$
- Similarly the force at node 2: $F_2 = k(\delta_2 - \delta_1)$

$$F_1 = k(\delta_1 - \delta_2) = -F_2$$

- We can re-write the above equations in matrix form as:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

The direct approach



$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

Load vector

Stiffness matrix

Displacement vector

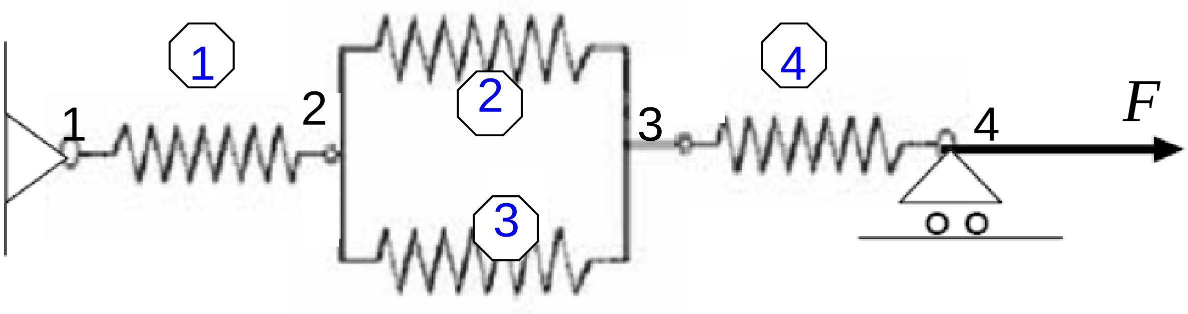
- We can generalize this equation for a linear spring as:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

k_{ij}: force on ith node induced by a unit displacement in the jth node

Direct method

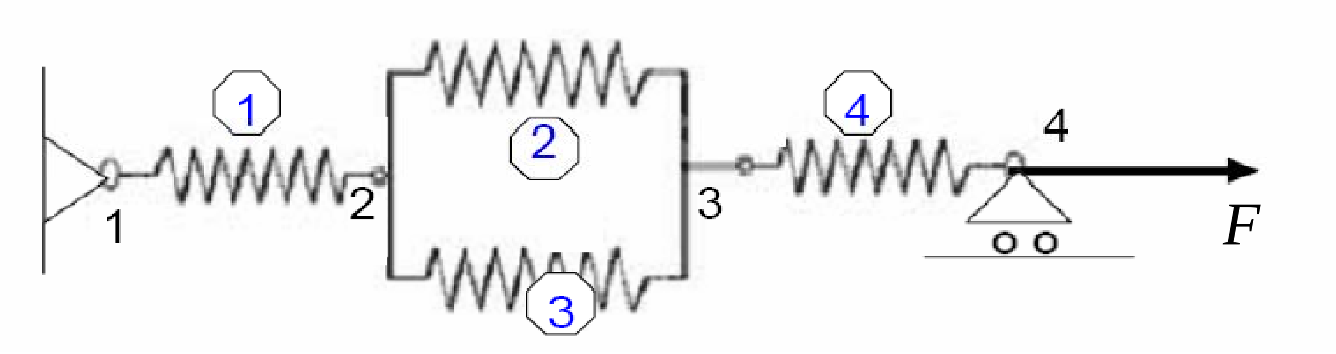
- Let us consider the analysis of the following loaded system of linear springs (each with 2 nodes)



- This system has 4 linear spring elements and there are 4 global nodes.
- Node 1 is fixed but node 4 can slide under the (known) applied load *F*.

We are interested to compute the nodal displacements δ_1, δ_2, δ_3, δ_4

Direct method



- We know the behavior of each `element', e.g.

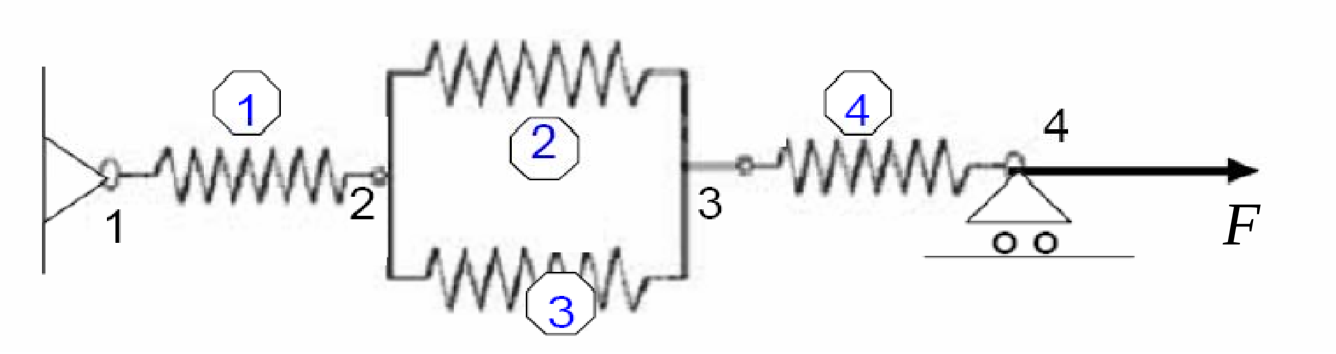
$$\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} \delta_1^{(1)} \\ \delta_2^{(1)} \end{Bmatrix}$$

Element	Local Node	Global Node
1	1	1
	2	2

$\delta_1^{(1)} \longrightarrow$ Displacement of local node 1 of element 1

$\delta_2^{(1)} \longrightarrow$ Displacement of local node 2 of element 1

Direct method



- For `element' 2:

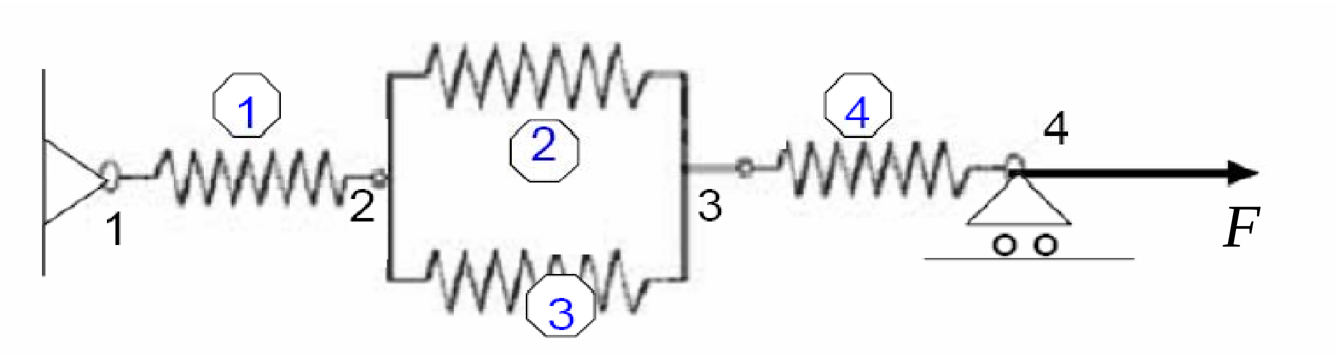
$$\begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} \delta_1^{(2)} \\ \delta_2^{(2)} \end{Bmatrix}$$

Element	Local Node	Global Node
2	1	2
	2	3

$\delta_1^{(2)} \longrightarrow$ Displacement of local node 1 of element 2

$\delta_2^{(2)} \longrightarrow$ Displacement of local node 2 of element 2

Direct method



- For `element' 3:

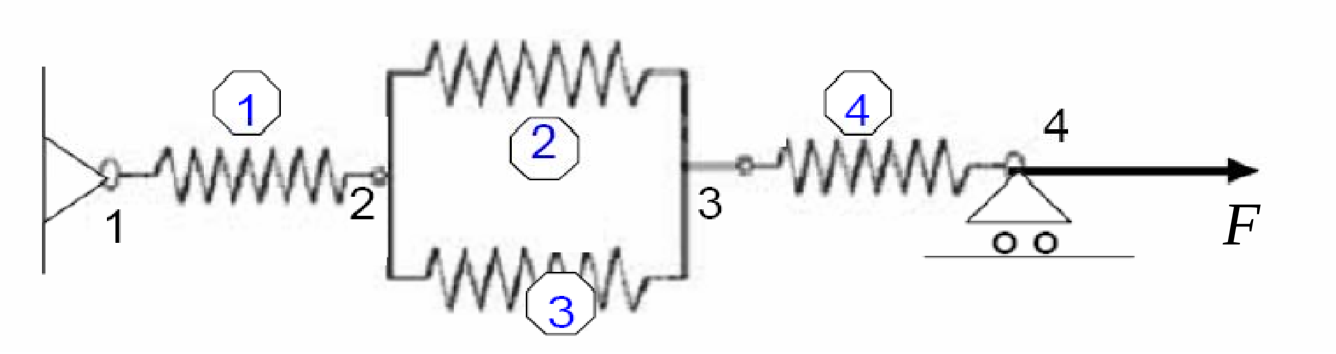
$$\begin{Bmatrix} F_1^{(3)} \\ F_2^{(3)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \begin{Bmatrix} \delta_1^{(3)} \\ \delta_2^{(3)} \end{Bmatrix}$$

Element	Local Node	Global Node
3	1	2
	2	3

$\delta_1^{(3)} \longrightarrow$ Displacement of local node 1 of element 3

$\delta_2^{(3)} \longrightarrow$ Displacement of local node 2 of element 3

Direct method



- For `element' 4:

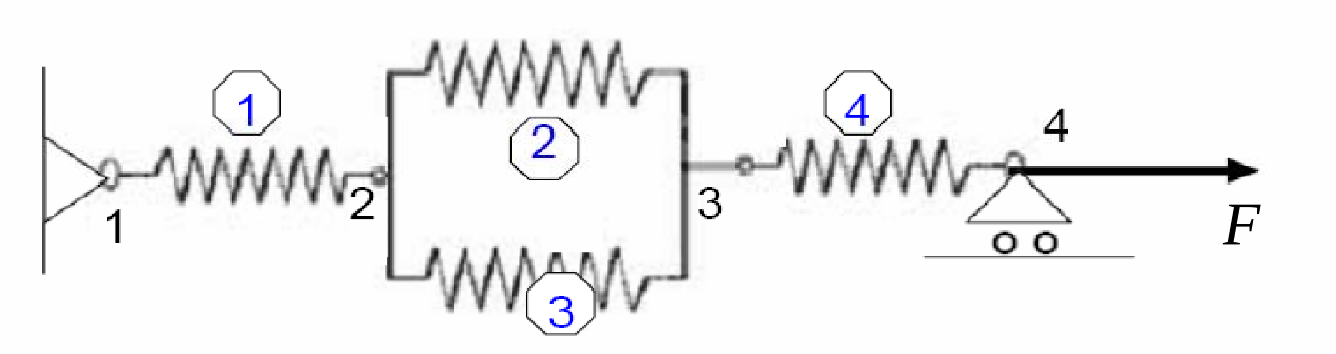
$$\begin{Bmatrix} F_1^{(4)} \\ F_2^{(4)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(4)} & k_{12}^{(4)} \\ k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} \delta_1^{(4)} \\ \delta_2^{(4)} \end{Bmatrix}$$

Element	Local Node	Global Node
4	1	3
	2	4

$\delta_1^{(4)} \longrightarrow$ Displacement of local node 1 of element 4

$\delta_2^{(4)} \longrightarrow$ Displacement of local node 2 of element 4

Connectivity matrix



- Connectivity matrix T:

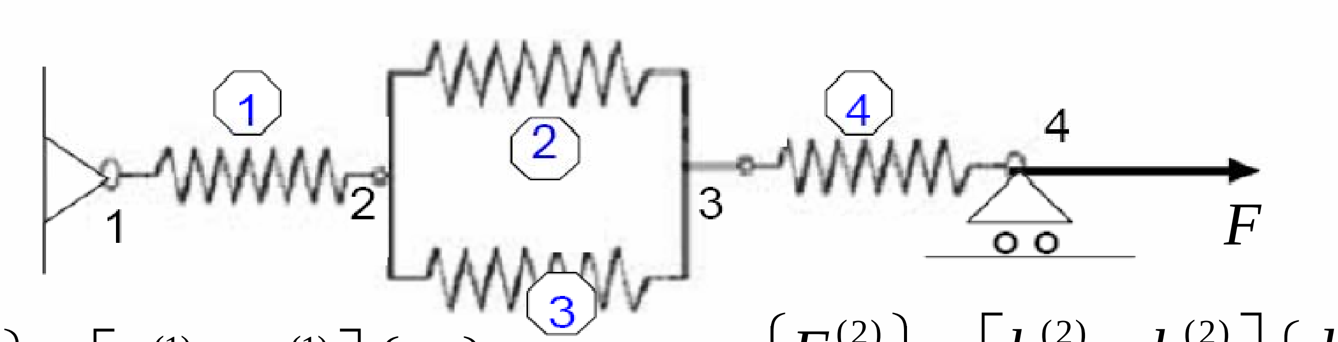
Global nodes of element 2

$$T = \begin{bmatrix} 1 & \boxed{\begin{matrix} 2 \\ 3 \end{matrix}} & 2 & 3 \\ 2 & & 3 & 4 \end{bmatrix}$$

e=2

One column for each element

Summary



$$\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

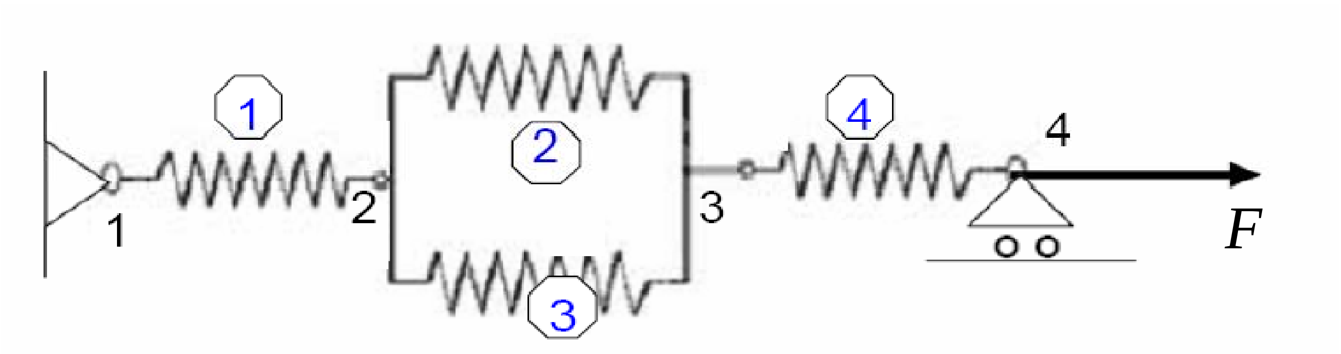
$$\begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1^{(3)} \\ F_2^{(3)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1^{(4)} \\ F_2^{(4)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(4)} & k_{12}^{(4)} \\ k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_3 \\ d_4 \end{Bmatrix}$$

- We use d_1, d_2, d_3, d_4 from now on to denote the global degrees of freedom.
- Let us re-write each of these equations to include in the displacement vector all nodal displacements of our system

Local to global transformation

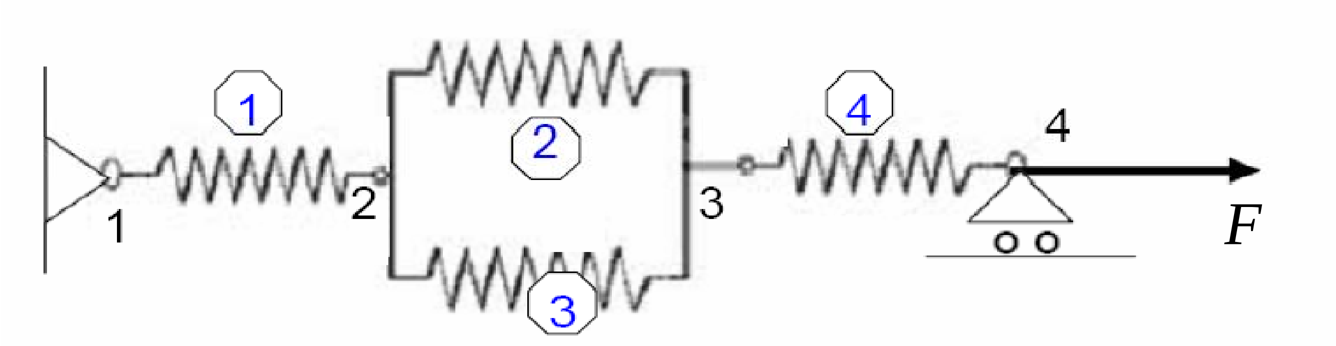


$$\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \longrightarrow \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

Global node displacements

$$\begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_1^{(2)} \\ F_2^{(2)} \\ 0 \end{Bmatrix}$$

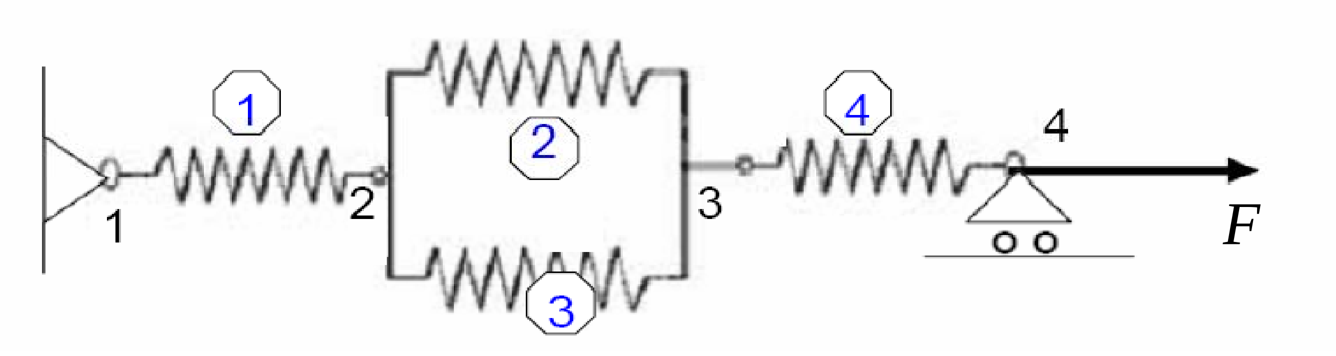
Local to global transformation



$$\begin{Bmatrix} F_1^{(3)} \\ F_2^{(3)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \end{Bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_{11}^{(3)} & k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(3)} & k_{22}^{(3)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_1^{(3)} \\ F_2^{(3)} \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1^{(4)} \\ F_2^{(4)} \end{Bmatrix} = \begin{bmatrix} k_{11}^{(4)} & k_{12}^{(4)} \\ k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_3 \\ d_4 \end{Bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_1^{(4)} \\ F_2^{(4)} \end{Bmatrix}$$

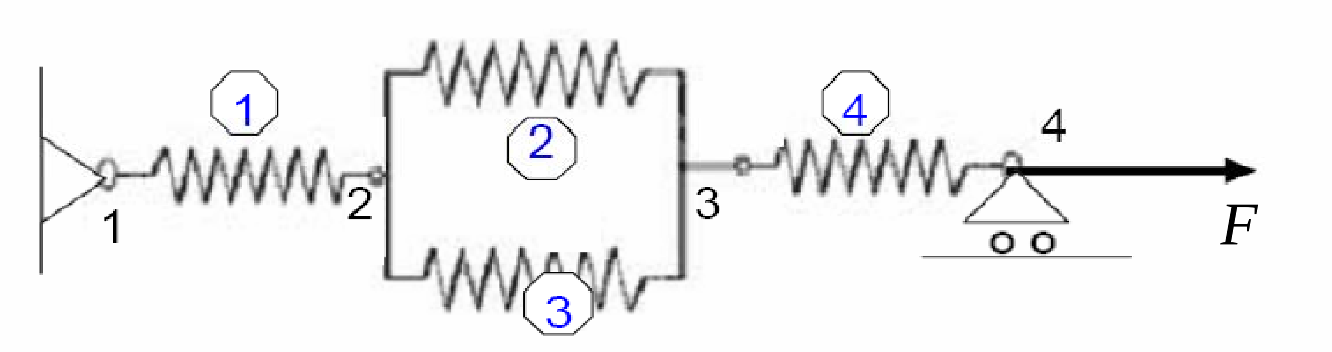
Assembly: $[K]=\sum_e [K]^{(e)},[F]=\sum_e [F]^{(e)}$



$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)}+k_{11}^{(3)} & k_{12}^{(2)}+k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(2)}+k_{21}^{(3)} & k_{22}^{(2)}+k_{22}^{(3)}+k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)}+F_1^{(2)}+F_1^{(3)} \\ F_2^{(2)}+F_2^{(3)}+F_1^{(4)} \\ F_2^{(4)} \end{Bmatrix}$$

- What contributes to the global stiffness component *K_{ij}, i.j=1,4?*
  - ✓Elements between nodes i and j.

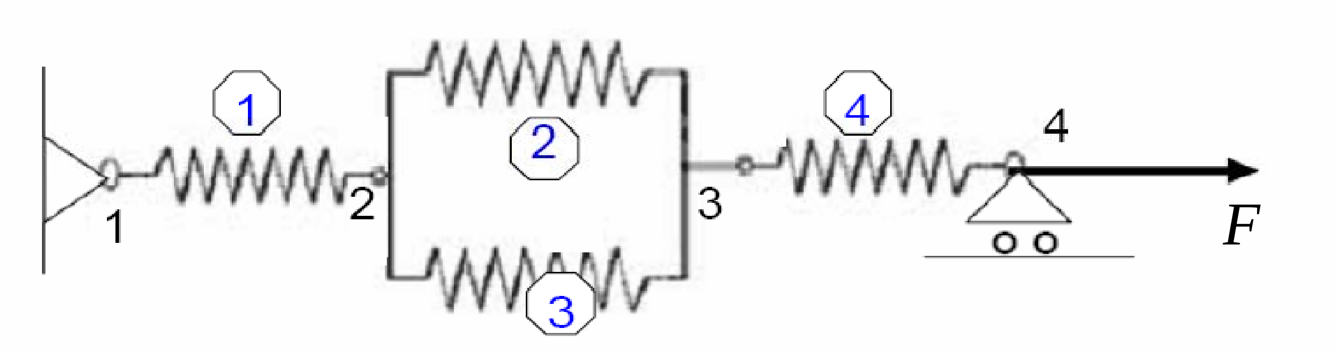
Assembly: $[K]=\sum_e [K]^{(e)}, [F]=\sum_e [F]^{(e)}$



$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)}+k_{11}^{(3)} & k_{12}^{(2)}+k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(2)}+k_{21}^{(3)} & k_{22}^{(2)}+k_{22}^{(3)}+k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)}+F_1^{(2)}+F_1^{(3)} \\ F_2^{(2)}+F_2^{(3)}+F_1^{(4)} \\ F_2^{(4)} \end{Bmatrix}$$

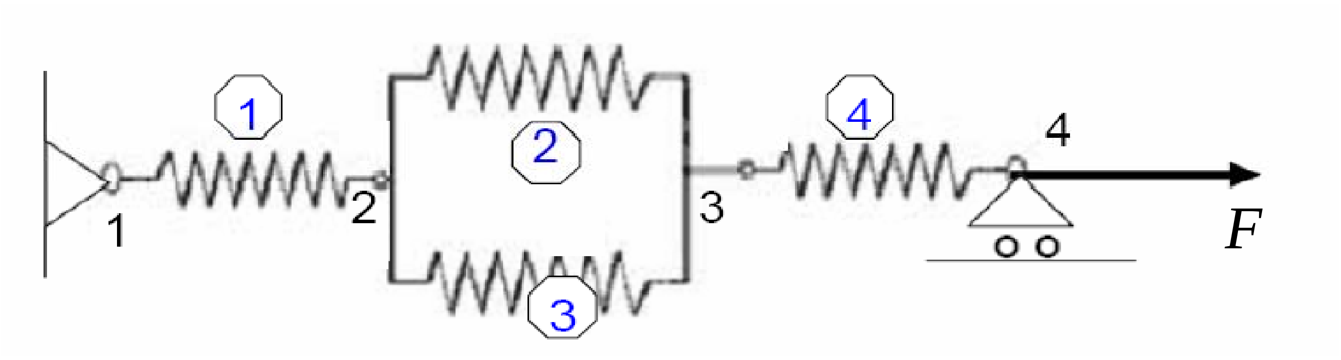
- What contributes to the global stiffness component K_{ii}, $i=1,4$?
  - ✓Elements that share node *i*.

Assembly: $[K]=\sum_{e}[K]^{(e)},[F]=\sum_{e}[F]^{(e)}$



$$\underbrace{\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)}+k_{11}^{(3)} & k_{12}^{(2)}+k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(2)}+k_{21}^{(3)} & k_{22}^{(2)}+k_{22}^{(3)}+k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix}}_{\text{Global stiffness matrix}} \underbrace{\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}}_{\substack{\text{Displacement} \\ \text{vector of global} \\ \text{nodes}}} = \underbrace{\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)}+F_1^{(2)}+F_1^{(3)} \\ F_2^{(2)}+F_2^{(3)}+F_1^{(4)} \\ F_2^{(4)} \end{Bmatrix}}_{\text{Global load vector}}$$

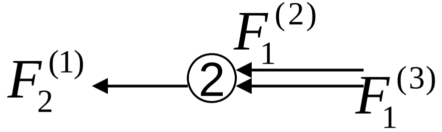
Assembly: $[K]=\sum_e [K]^{(e)},[F]=\sum_e [F]^{(e)}$

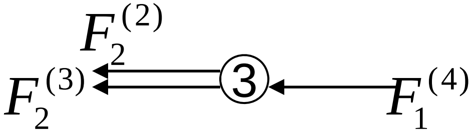


$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)}+k_{11}^{(3)} & k_{12}^{(2)}+k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(2)}+k_{21}^{(3)} & k_{22}^{(2)}+k_{22}^{(3)}+k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)}+F_1^{(2)}+F_1^{(3)} \\ F_2^{(2)}+F_2^{(3)}+F_1^{(4)} \\ F_2^{(4)} \end{Bmatrix}$$

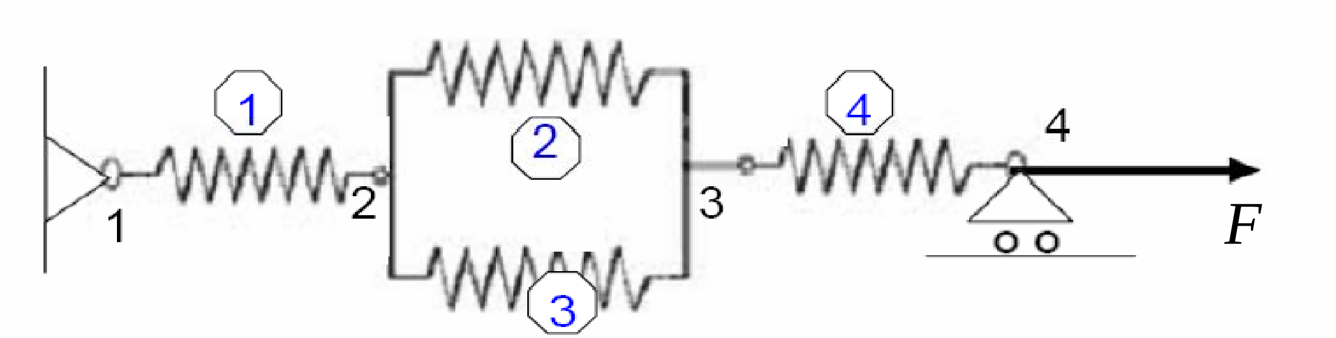
- What is the total force acting on nodes 2 and 3?

$$F_2^{(1)}+F_1^{(2)}+F_1^{(3)}=0 \qquad F_2^{(2)}+F_2^{(3)}+F_1^{(4)}=0$$



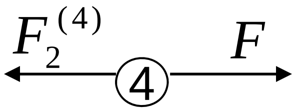


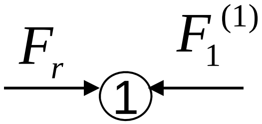
Assembly: $[K]=\sum_e [K]^{(e)}, [F]=\sum_e [F]^{(e)}$



$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)}+k_{11}^{(3)} & k_{12}^{(2)}+k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(2)}+k_{21}^{(3)} & k_{22}^{(2)}+k_{22}^{(3)}+k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ 0 \\ 0 \\ F_2^{(4)} \end{Bmatrix}$$

- How about $F_1^{(1)}$ and $F_2^{(4)}$?

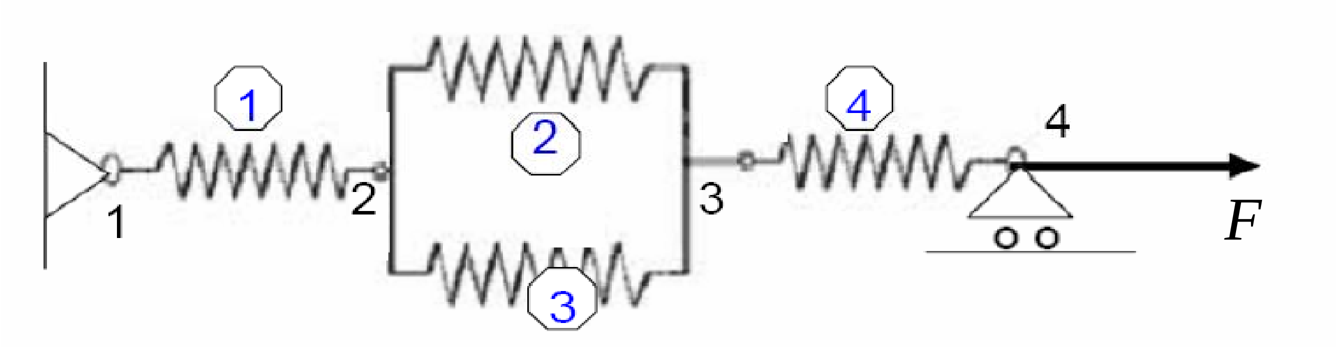




$F_2^{(4)} = F = applied\ force\,(known)$

$F_1^{(1)} = F_r = reaction\ force\,(unknown)$

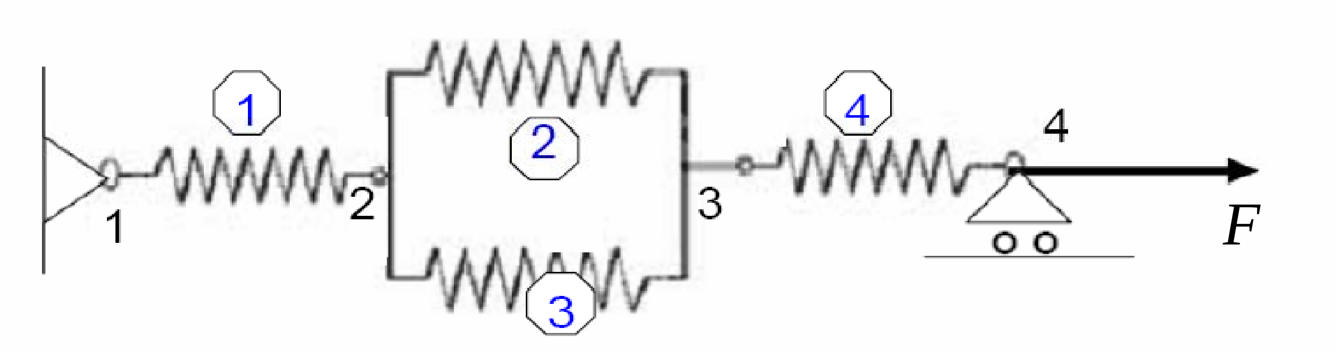
Assembly: $[K]=\sum_e [K]^{(e)},[F]=\sum_e [F]^{(e)}$



$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)}+k_{11}^{(3)} & k_{12}^{(2)}+k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(2)}+k_{21}^{(3)} & k_{22}^{(2)}+k_{22}^{(3)}+k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} F_r \\ 0 \\ 0 \\ F \end{Bmatrix}$$

- We have 4 equations with 5 unknowns: d_1,d_2,d_3,d_4,F_r
- We have not yet used the boundary condition: $d_1=0$

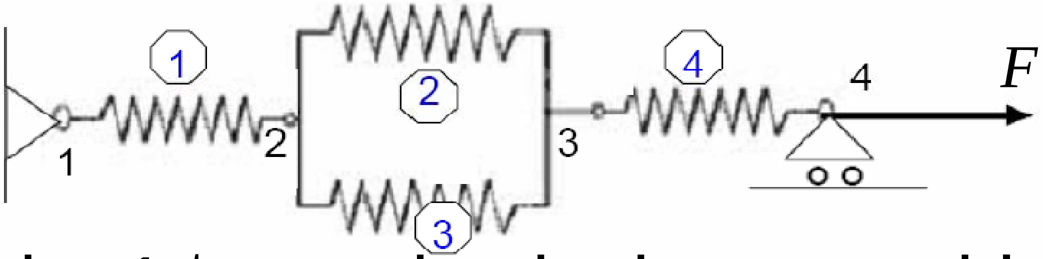
Displacement calculation



$$\begin{bmatrix} k_{22}^{(1)}+k_{11}^{(2)}+k_{11}^{(3)} & k_{12}^{(2)}+k_{12}^{(3)} & 0 \\ k_{21}^{(2)}+k_{21}^{(3)} & k_{22}^{(2)}+k_{22}^{(3)}+k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix}$$

- This 3x3 system of eqs can be solved for: d_2, d_3, d_4
- Once you do that, how do you compute the reaction force at node 1?

Reaction force calculation

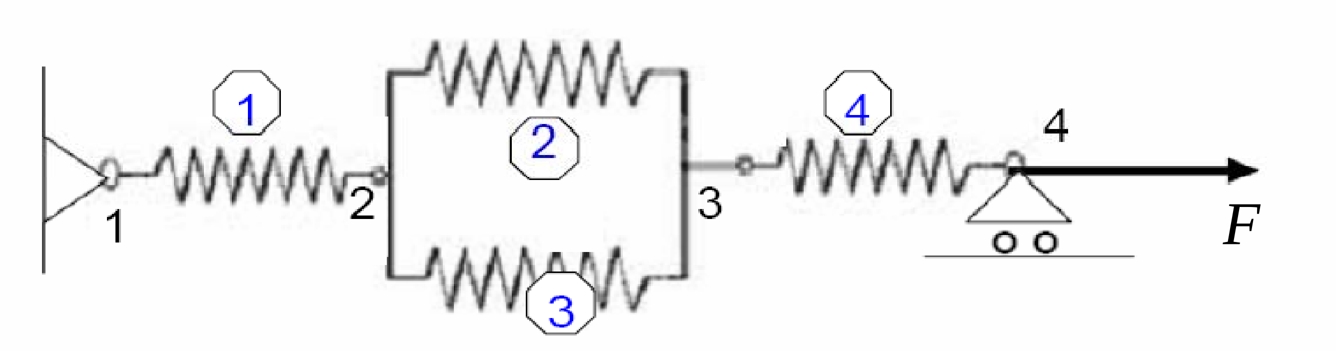


- Return to the 1st equation in the assembled system before the application of the boundary condition:

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)}+k_{11}^{(3)} & k_{12}^{(2)}+k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(2)}+k_{21}^{(3)} & k_{22}^{(2)}+k_{22}^{(3)}+k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} F_r \\ 0 \\ 0 \\ F \end{Bmatrix}$$

$$\begin{bmatrix} k_{11}^{(1)} \; k_{12}^{(1)} \; 0 \; 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = F_r \Rightarrow F_r = k_{11}^{(1)} d_1 + k_{12}^{(1)} d_2 = k_{12}^{(1)} d_2$$

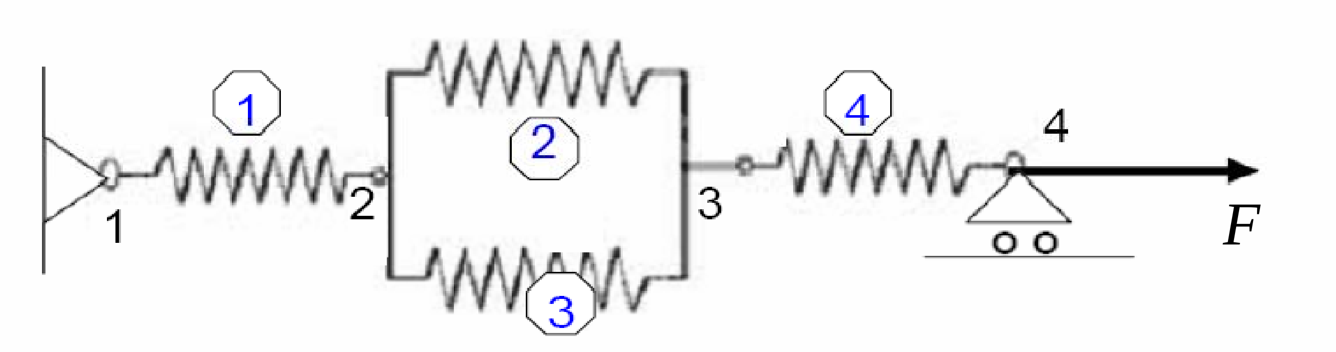
Non-zero displacement at node 1: $\delta_1 = \alpha$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} + k_{11}^{(3)} & k_{12}^{(2)} + k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(2)} + k_{21}^{(3)} & k_{22}^{(2)} + k_{22}^{(3)} + k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} \alpha \\ 0 \\ 0 \\ F \end{Bmatrix} \Rightarrow 4x4$$

system of Eqs

Non-zero displacement at node 1: $\delta_1 = \alpha$

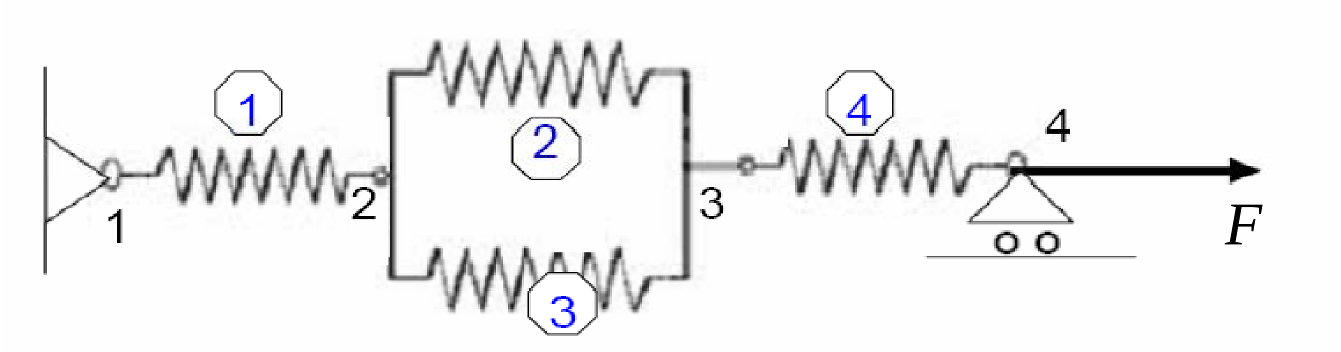


$$\begin{bmatrix} k_{22}^{(1)} + k_{11}^{(2)} + k_{11}^{(3)} & k_{12}^{(2)} + k_{12}^{(3)} & 0 \\ k_{21}^{(2)} + k_{21}^{(3)} & k_{22}^{(2)} + k_{22}^{(3)} + k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} -k_{21}^{(1)}\alpha \\ 0 \\ F \end{Bmatrix} \Rightarrow 3x3$$ system of Eqs

- The reaction force can be computed as before:

$$\begin{bmatrix} k_{11}^{(1)} \; k_{12}^{(1)} \; 0 \; 0 \end{bmatrix} \begin{Bmatrix} \alpha \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = F_r \Rightarrow F_r = k_{11}^{(1)}\alpha + k_{12}^{(1)} d_2$$

Consider $k_{ii}^{(e)} = k,\, k_{ij}^{(e)} = k_{ji}^{(e)} = -k, \text{for all elements } e, \text{i.e.}\, K^{(e)} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$



$$\begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix} \Rightarrow \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F/k \\ 3F/2k \\ 5F/2k \end{Bmatrix}$$

Symmetric stiffness

- The reaction force can then be computed as:

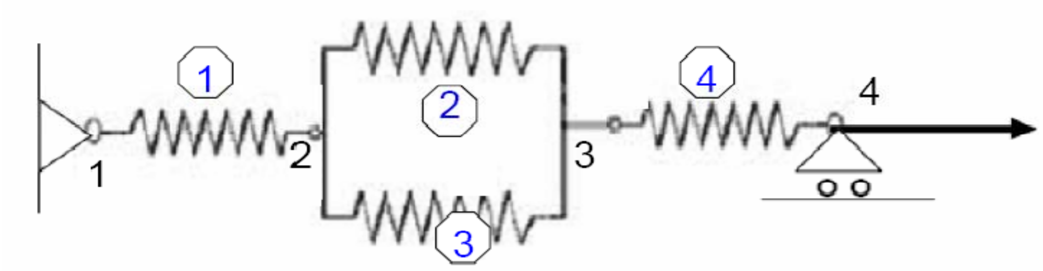
$$F_r = k_{12}^{(1)} d_2 = -k \, {}^{F}\!\big/_{k} = -F\,!$$

Revisiting the assembly process

$$[K] = \sum_e [K]^{(e)}, [F] = \sum_e [F]^{(e)}$$

- In the equations above, we imply that the element stiffness $[K]^{(e)}$ and load vectors $[F]^{(e)}$ are already written in the expanded global node format.
- How do we write the above assembly process if we want to use element stiffness $[K]'^{(e)}$ expressed in the local node format?

Revisiting the assembly process



- Recall that e.g. element 1 in local format:

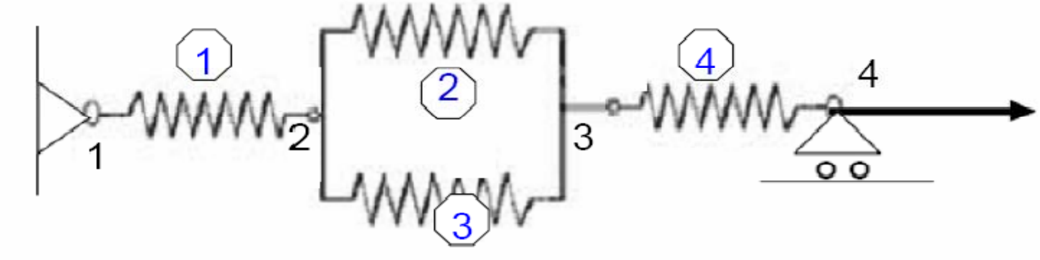
$$\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{Bmatrix} = \underbrace{\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix}}_{[K'^{(1)}]} \underbrace{\begin{Bmatrix} \delta_1^{(1)} \\ \delta_2^{(1)} \end{Bmatrix}}_{\{d^{(1)}\}} \equiv [K'^{(1)}]\{d^{(1)}\}$$

- We can write the following transformations:

$$\{d^{(1)}\} \equiv \begin{Bmatrix} \delta_1^{(1)} \\ \delta_2^{(1)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = [L^{(1)}]\{d\}, \quad \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \end{Bmatrix} = [L^{(1)}]^T \{F^{(1)}\}$$

- Note that the matrix $[L^{(1)}]^T$ scatters the nodal forces into the global nodal form

Revisiting the assembly process



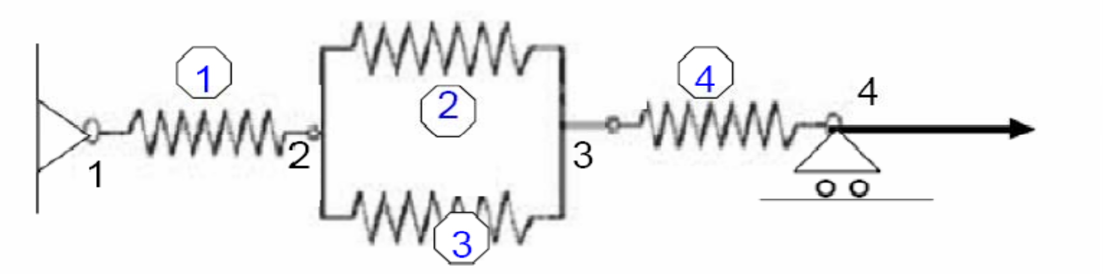
- But we have seen that from equilibrium of each node:

$$\begin{Bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ F_1^{(2)} \\ F_2^{(2)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ F_1^{(3)} \\ F_2^{(3)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ F_1^{(4)} \\ F_2^{(4)} \end{Bmatrix} = \begin{Bmatrix} F_r \\ 0 \\ 0 \\ F \end{Bmatrix} \equiv \{f\}$$

- If we call the applied external force vector simply $\{f\}$,we can summarize the above as:

$$\sum_e \underbrace{[L^{(e)}]^T}_{4x2} \underbrace{\{F^{(e)}\}}_{2x1} = \underbrace{\{f\}}_{4x1}$$

Revisiting the assembly process



- Return to the element equations

$$\{F^{(e)}\}=\begin{Bmatrix}F_1^{(e)}\\ F_2^{(e)}\end{Bmatrix}=\begin{bmatrix}k_{11}^{(e)} & k_{12}^{(e)}\\ k_{21}^{(e)} & k_{22}^{(e)}\end{bmatrix}\begin{Bmatrix}\delta_1^{(e)}\\ \delta_2^{(e)}\end{Bmatrix}\equiv[K'^{(e)}]\{d^{(e)}\}$$

and the transformations:

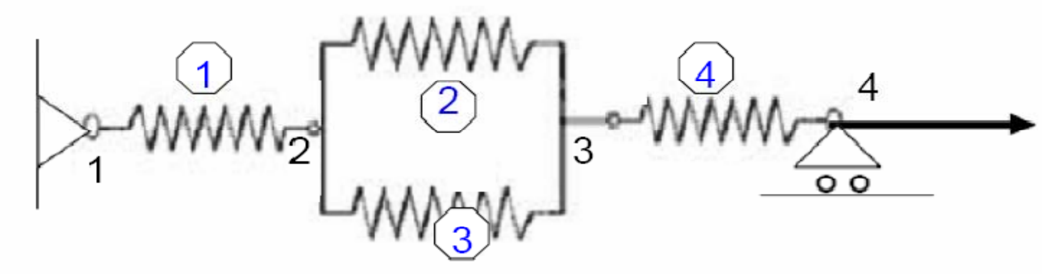
$$\underbrace{\begin{Bmatrix}\delta_1^{(e)}\\ \delta_2^{(e)}\end{Bmatrix}}_{\{d^{(e)}\}}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{Bmatrix}d_1\\ d_2\\ d_3\\ d_4\end{Bmatrix}=[L^{(e)}]\{d\}$$

$$\sum_{e=1}^{4}[L^{(e)}]^T\{F^{(e)}\}=\{f\}$$

- Combining these 3 Eqs gives: $\sum_e [L^{(e)}]^T[K'^{(e)}]\{d^{(e)}\}=\{f\}\Rightarrow$

$$\underbrace{\sum_e [L^{(e)}]^T[K'^{(e)}][L^{(e)}]}_{K}\{d\}=\{f\}\Rightarrow \quad \{f\}=[K]\{d\}, \quad where \quad \underbrace{[K]}_{4x4}=\sum_e \underbrace{[L^{(e)}]^T}_{4x2}\underbrace{[K'^{(e)}]}_{2x2}\underbrace{[L^{(e)}]}_{2x4}$$

Revisiting the assembly process



- Let us verify $[K^{(1)}]=[L^{(1)}]^T[K'^{(1)}][L^{(1)}]$

Element stiffness in global node format

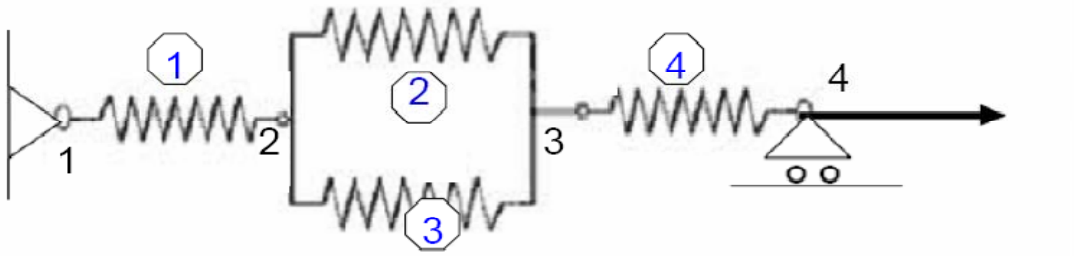
Stiffness in local node format

Transformation matrix from local to global node format

$$[K^{(1)}]=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}\begin{bmatrix}k_{11}^{(1)}&k_{12}^{(1)}\\k_{21}^{(1)}&k_{22}^{(1)}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}=\begin{bmatrix}k_{11}^{(1)}&k_{12}^{(1)}\\k_{21}^{(1)}&k_{22}^{(1)}\\0&0\\0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}=\begin{bmatrix}k_{11}^{(1)}&k_{12}^{(1)}&0&0\\k_{21}^{(1)}&k_{22}^{(1)}&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

- This is indeed what we used before!

Revisiting the assembly process

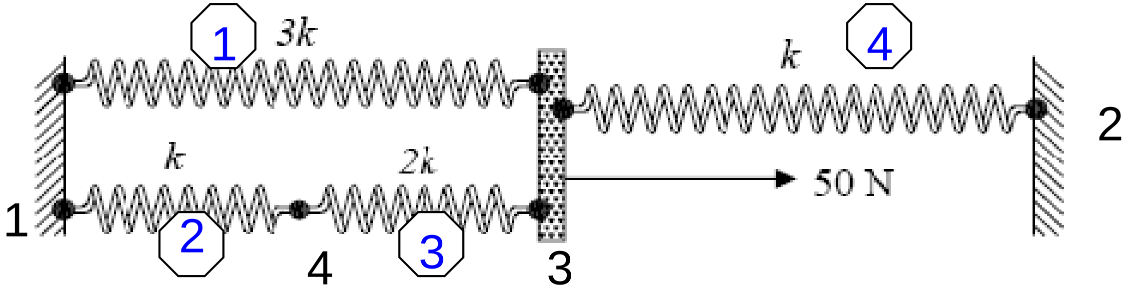


$$\sum_e [L^{(e)}]^T \{F^{(e)}\} = \{f\} \qquad [K] = \sum_e \underbrace{[L^{(e)}]^T [K^{'(e)}][L^{(e)}]}_{[K^{(e)}]}$$

$$[K]\{d\} = \{f\}$$

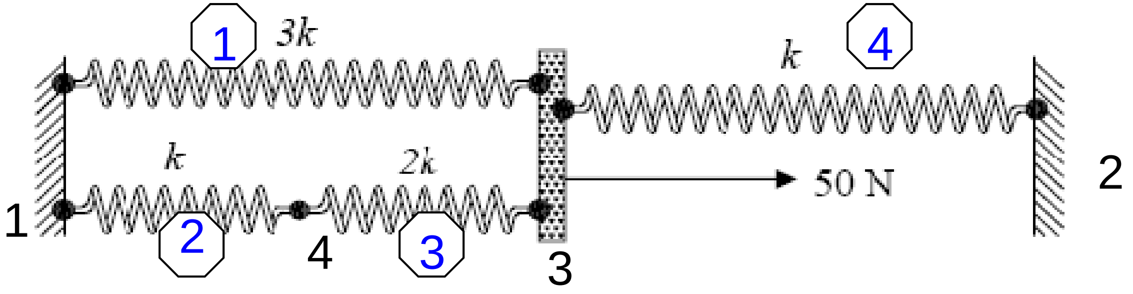
- We remind you that $[K^{(e)}]$ is the e-element stiffness in global nodal notation and $[K^{'(e)}]$ the e-element stiffness in local element nodal notation.

Another example



- For the spring system above, compute the
  - global stiffness and force vector
  - partition the system and solve for the nodal displacements
  - compute the reaction forces

An example

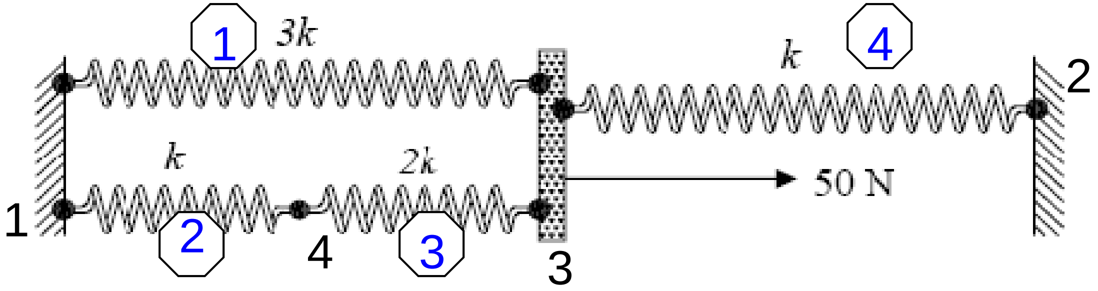


$$[K'^{(1)}] = k\begin{array}{cc} \begin{array}{cc}1 & \quad 3\end{array} & \\ \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix} & \begin{array}{c}1\\ 3\end{array}\end{array} \qquad [K'^{(2)}] = k\begin{array}{cc} \begin{array}{cc}1 & \quad 4\end{array} & \\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} & \begin{array}{c}1\\ 4\end{array}\end{array}$$

$$[K'^{(3)}] = k\begin{array}{cc} \begin{array}{cc}4 & \quad 3\end{array} & \\ \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix} & \begin{array}{c}4\\ 3\end{array}\end{array} \qquad [K'^{(4)}] = k\begin{array}{cc} \begin{array}{cc}3 & \quad 2\end{array} & \\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} & \begin{array}{c}3\\ 2\end{array}\end{array}$$

$$[K] = \begin{array}{cc} \begin{array}{cccc}1 & 2 & 3 & 4\end{array} & \\ \begin{bmatrix}3+1 & 0 & -3 & -1\\ 0 & 1 & -1 & 0\\ -3 & -1 & 3+2+1 & -2\\ -1 & 0 & -2 & 1+2\end{bmatrix} & \begin{array}{c}1\\ 2\\ 3\\ 4\end{array}\end{array}$$

An example

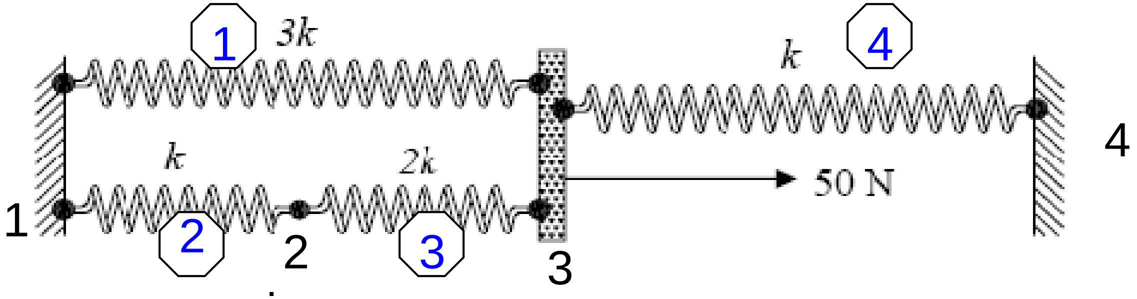


$$k\left[\begin{array}{cc|cc} 4 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ \hline -3 & -1 & 6 & -2 \\ -1 & 0 & -2 & 3 \end{array}\right]\left\{\begin{array}{c} d_1 \\ d_2 \\ \hline d_3 \\ d_4 \end{array}\right\}=\left\{\begin{array}{c} r_1 \\ r_2 \\ 50Nt \\ 0 \end{array}\right\}$$

- We partition and apply BCs: $d_1 = d_2 = 0$

$$k\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}\begin{Bmatrix} d_3 \\ d_4 \end{Bmatrix}=\begin{Bmatrix} 50Nt \\ 0 \end{Bmatrix}\Rightarrow\begin{Bmatrix} d_3 \\ d_4 \end{Bmatrix}=\frac{1}{k}\begin{Bmatrix} 10.7143 \\ 7.1429 \end{Bmatrix}Nt$$

An example



$$
k\left[\begin{array}{cc:cc} 4 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ \hdashline -3 & -1 & 6 & -2 \\ -1 & 0 & -2 & 3 \end{array}\right]\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ 50Nt \\ 0 \end{Bmatrix}
$$

- Compute the reaction forces:

$$
k\begin{bmatrix} -3 & -1 \\ -1 & 0 \end{bmatrix}\begin{Bmatrix} d_3 \\ d_4 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} \Rightarrow \qquad \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} -39.286 \\ -10.714 \end{Bmatrix} Nt
$$