MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

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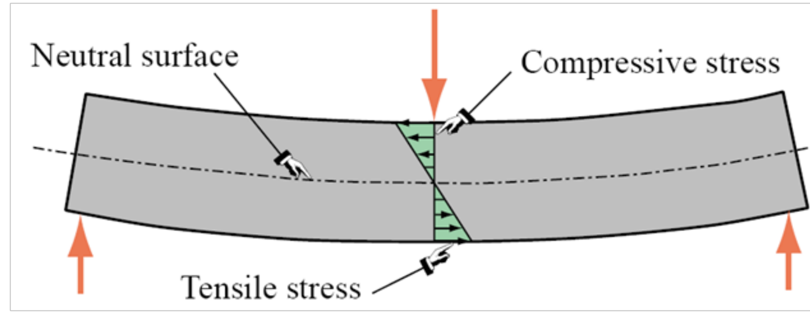
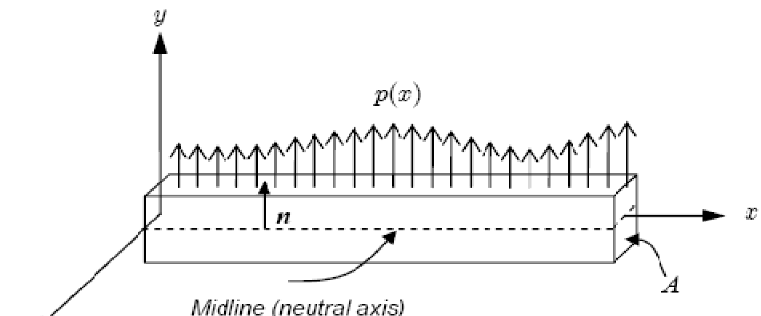
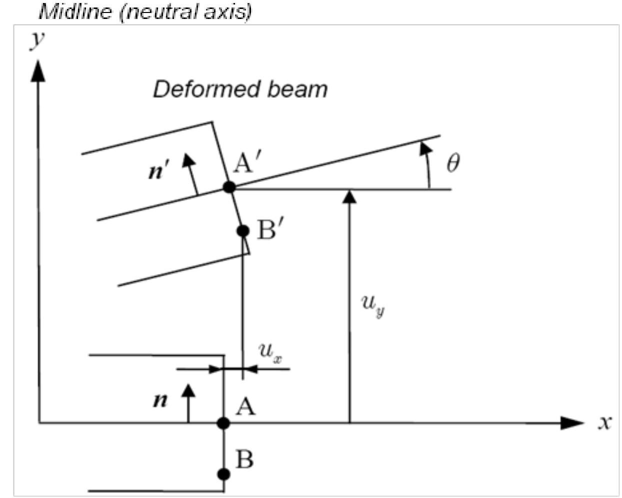
A refresher on beam bending

- Beams are different from truss structures in that they are designed to resist transverse loads.
- The tranverse loads are transported to the supports of the beam via extensional action.
- There are several beam models depending on the assumptions employed. We herein consider the Bernoulli-Euler beam theory usually introduced in introductory statics courses.
- We assume that normals to the middle line of the beam remain straight and normal.
- This will allow us to approximate the displacements u_x, u_y at a given point.





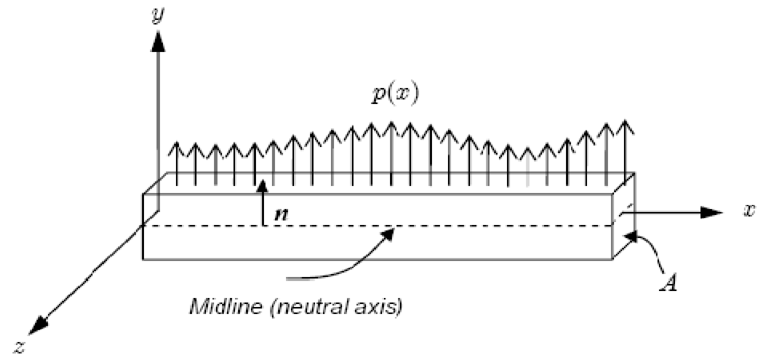


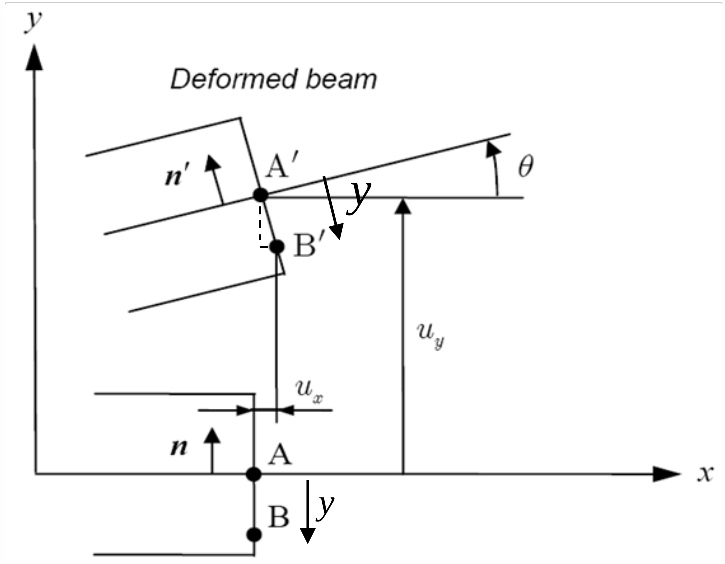
Bernoulli-Euler beam theory

- The x-component of the displacement u_x through the depth of the beam is given as:

$$u_x = -y \sin\theta(x)$$

where $\theta(x)$ is the rotation of the middle line (positive counterclockwise) at x and y is the distance from the middle line.





- We assume that $u_y(x,y) = u_y(x)$.
- For $\theta(x)$ small, $\sin\theta = \theta$,

$$\theta = \frac{du_y(x)}{dx}$$

Thus we conclude:

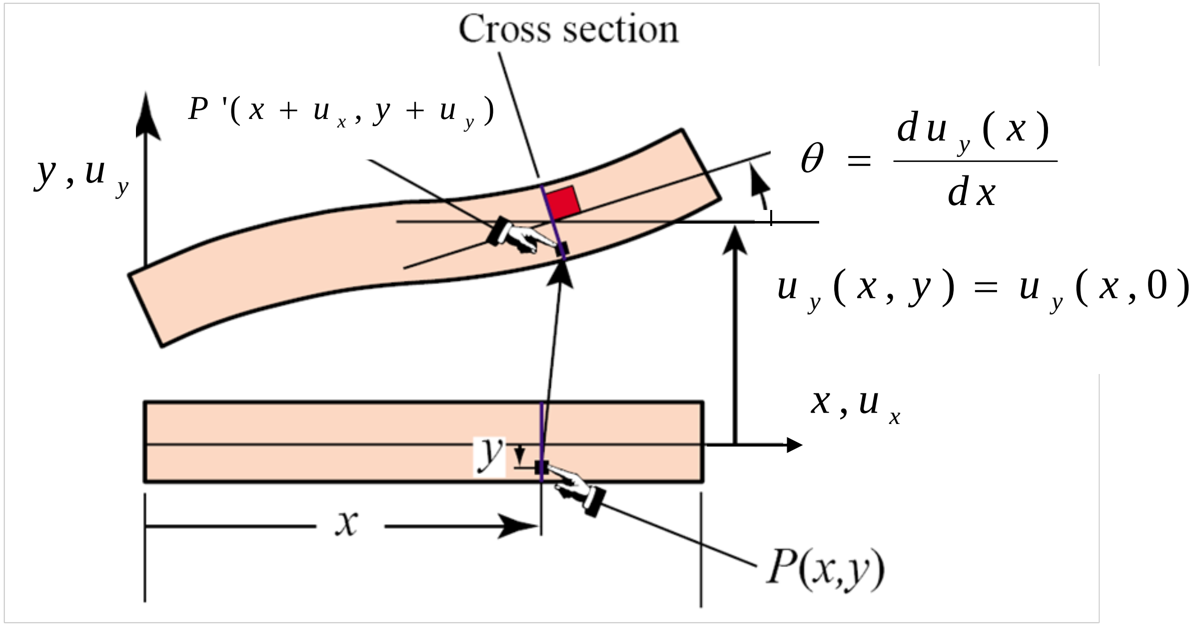
$$u_x = -y\frac{du_y}{dx} \Rightarrow \varepsilon_x = \frac{du_x}{dx} = -y\frac{d^2u_y}{dx^2} = -y\kappa$$

curvature

Displacements im Bernoulli-Euler Beam model

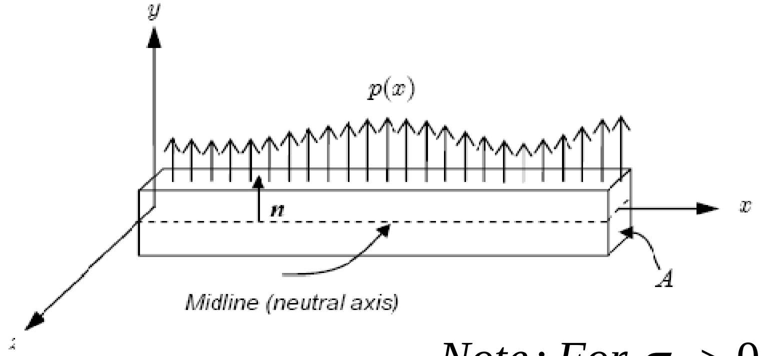
- In summary, the following displacements are considered:

$$\begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} -y\dfrac{du_y(x)}{dx} \\ u_y(x) \end{bmatrix} = \begin{bmatrix} -y\theta \\ u_y(x) \end{bmatrix}$$



Stress and moment calculation

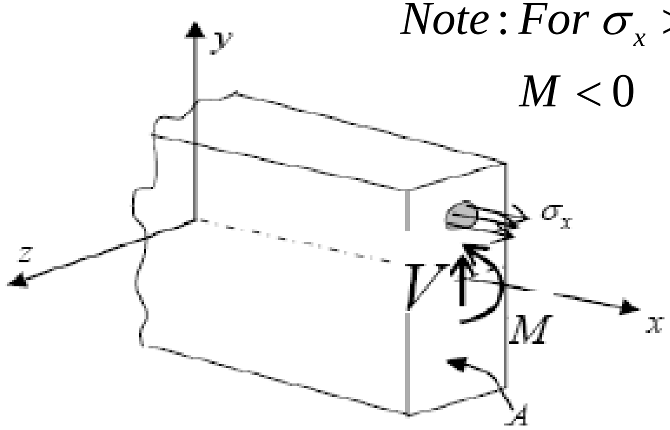
$$\varepsilon_x = -y\frac{d^2u_y}{dx^2}$$



- From Hooke's law, we can compute the axial stress as:

$$\sigma_x = E\varepsilon_x = -Ey\frac{d^2u_y}{dx^2}$$

$$Note: For\ \sigma_x > 0,\quad M < 0$$



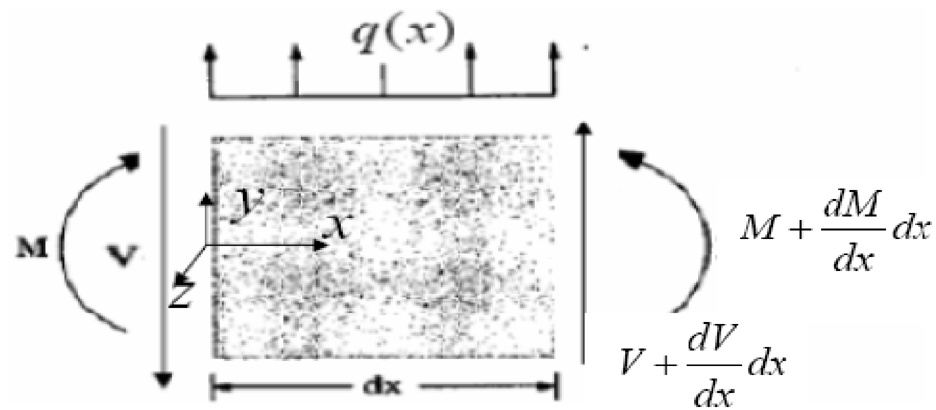
- With integration, we can now compute the bending moment as follows:

$$M(x) = -\int_A \sigma_x y dA = \int_A Ey\frac{d^2u_y}{dx^2} y dA = E\int_A y^2 dA \frac{d^2u_y}{dx^2} = EI\frac{d^2u_y}{dx^2}$$

- Here *I* denotes the moment of inertia of the cross section:

$$I = \int_A y^2 dA$$

Moments on a differential beam element



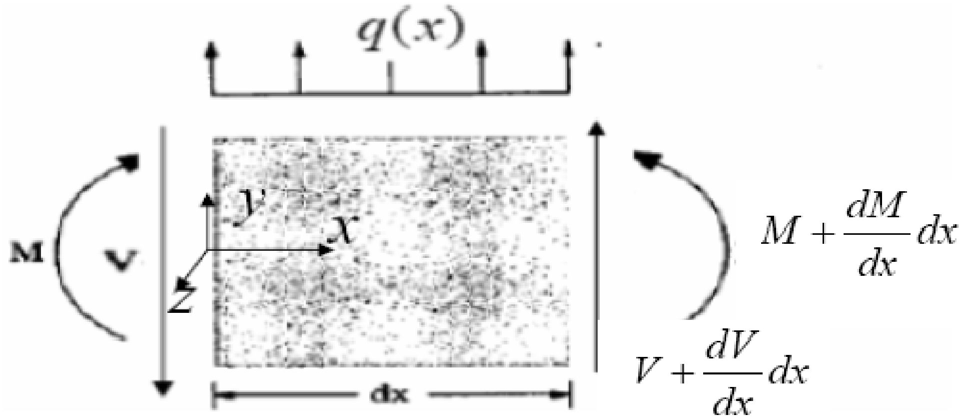
Sign convention
The *M* and *V's and load q as shown are positive (pay attention to planes with* positive and negative normal vectors)

- Let us apply balance of moments on this differential beam element around *x=y=0*

$$-M+(V+\frac{dV}{dx}dx)dx+\left(M+\frac{dM}{dx}dx\right)+\frac{dx}{2}q(x+\frac{dx}{2})dx=0$$

- From which we conclude that: $V=-\frac{dM}{dx}$

Forces on a differential beam element



- Let us apply balance of vertical forces on this differential beam element

$$\left(V+\frac{dV}{dx}dx\right)-V+q(x)dx=0$$

- From which we conclude that: $q=-\frac{dV}{dx}$

Differential equation for the beam

$$M(x) = EI\frac{d^2u_y}{dx^2}, V = -\frac{dM}{dx} = -\frac{d}{dx}\left(EI\frac{d^2u_y}{dx^2}\right),$$

$$q = -\frac{dV}{dx} = \frac{d^2}{dx^2}\left(EI\frac{d^2u_y}{dx^2}\right)$$

- The differential equation is 4th order for the vertical displacement u_y of the middle line.
- As a result, two boundary conditions are needed at each end!
- Variables that are conjugate in the sense of work (shear force *V* and u_y, moment *M* and rotation θ), cannot be both prescribed on the same boundary (same end of the beam).

$$\Gamma_V \cap \Gamma_u = 0, \Gamma_V \cup \Gamma_u = \Gamma$$

$$\Gamma_M \cap \Gamma_\theta = 0, \Gamma_M \cup \Gamma_\theta = \Gamma$$

Γ_V : *boundary with prescribed* V

Γ_u : *boundary with prescribed* u_y

Γ_M : *boundary with prescribed* M

Γ_θ : *boundary with prescribed* θ

Γ : *whole boundary* (*both ends*)

Boundary conditions

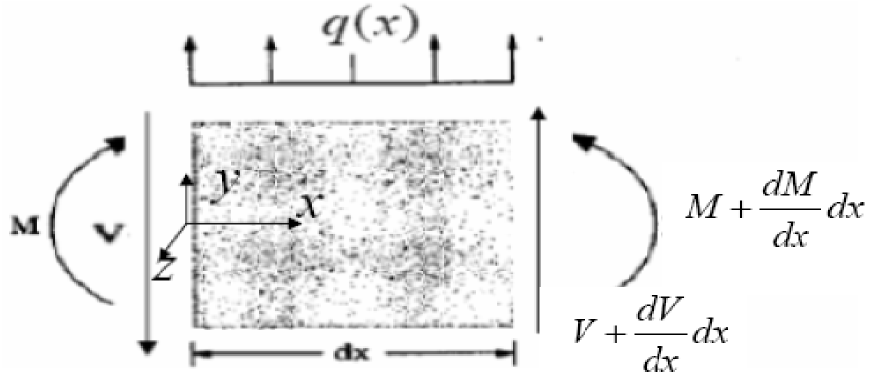
- These boundary conditions take the following forms:

$$u_y = \overline{u}_y \quad on \quad \Gamma_u,$$

$$\frac{du_y}{dx} = \overline{\theta} \quad on \quad \Gamma_\vartheta,$$

$$Mn \equiv EI \frac{d^2 u_y}{dx^2} n = \overline{M} \quad on \quad \Gamma_M,$$

$$Vn \equiv -\frac{d}{dx}\left(EI \frac{d^2 u_y}{dx^2} \right) n = \overline{V} \quad on \quad \Gamma_V.$$



- The given $\overline{M}$ and $\overline{V}$ are defined (our choice) positive when acting counterclockwise and in the positive *y*-direction, respectively.
- The normal $n = \pm 1$ is introduced in the last two conditions to maintain consistency with our sign convention for *V* and *M* (discussed further below).

Boundary conditions

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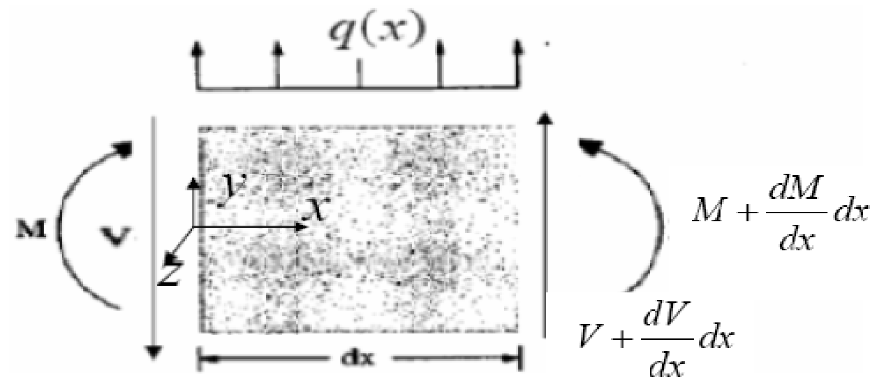
$$u_y = \bar{u}_y \quad on \quad \Gamma_u,$$

$$\frac{du_y}{dx} = \bar{\theta} \quad on \quad \Gamma_\vartheta,$$

$$Mn \equiv EI\frac{d^2u_y}{dx^2}n = \bar{M} \quad on \quad \Gamma_M,$$

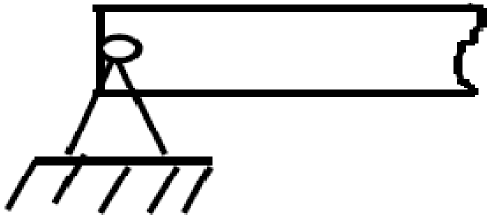
$$Vn \equiv -\frac{d}{dx}\left(EI\frac{d^2u_y}{dx^2}\right)n = \bar{V} \quad on \quad \Gamma_V.$$



- For example, if $\bar{M} > 0$ is prescribed on the right end (*n=1*), then $M \equiv EI\frac{d^2u_y}{dx^2} = \bar{M}.$ If $\bar{M} > 0$ is prescribed on the left end (*n=-1*), then: $M \equiv EI\frac{d^2u_y}{dx^2} = -\bar{M}.$ Similarly for $\bar{V}.$

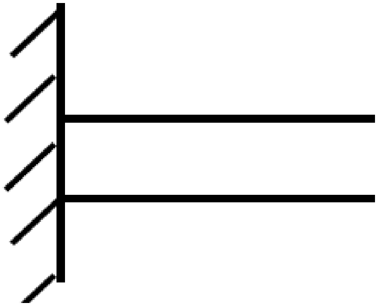
Boundary conditions for beams

- Free end with an applied load: $Mn = \overline{M} \;\; on \;\; \Gamma_M ,$ $Vn = \overline{V} \;\; on \;\; \Gamma_V .$
- Simple support:



$\overline{u}_y = 0 \;\; on \;\; \Gamma_u ,$ $\overline{M} = 0 \;\; on \;\; \Gamma_M .$

- Clamped support:



$\overline{u}_y = 0 \;\; on \;\; \Gamma_u ,$ $\overline{\theta} = 0 \;\; on \;\; \Gamma_\theta .$

Potential energy of a beam element: $P^e = U^e - W^e$

$$M(x) = EI\frac{d^2u_y}{dx^2}, V = -\frac{dM}{dx} = -\frac{d}{dx}\left(EI\frac{d^2u_y}{dx^2}\right),$$

$$q = -\frac{dV}{dx} = \frac{d^2}{dx^2}\left(EI\frac{d^2u_y}{dx^2}\right)$$

- Using these, let us compute the potential energy of `a beam element' of length L^e (to be defined shortly in more detail).

- Strain energy: $U^e = \int_{\Omega_e} \frac{E^e \varepsilon_x^2}{2} dV = \int_{\Omega_e} \frac{E^e (\kappa y)^2}{2} dA dx = \int_{L^e} \frac{E^e}{2} \underbrace{\int_{A^e} y^2 dA}_{I^e} \kappa^2 dx$

$$U^e = \int_{\Omega^e} \frac{E^e I^e}{2} \kappa^2 dx = \int_{\Omega^e} \frac{E^e I^e}{2} \left(\frac{d^2 u_y^e}{dx^2}\right)^2 dx$$

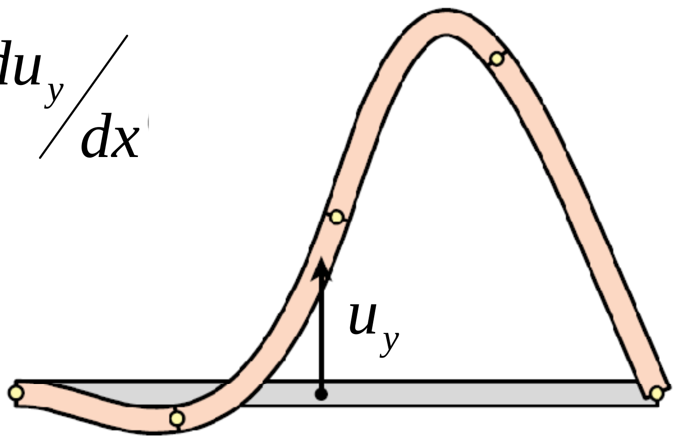
Notes:

- For this strain energy to make sense as an integral, we need to introduce an approximation (interpolation) for u_y^e such that $d^2u_y^e / dx^2$ is square integrable (**C¹ continuity**)
- From now-on, we simply denote with Ω^e the *x*-dimensional domain of a beam element e of length L^e (to be defined later in more detail)

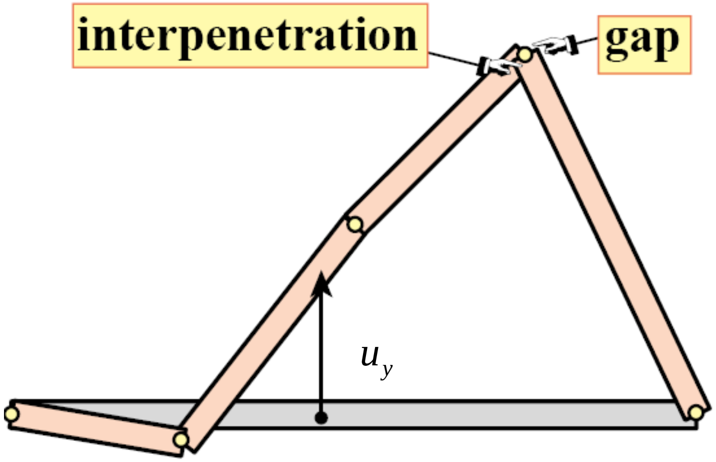
- External work: $W = \sum_e \left\{ \int_{\Omega^e} q(x) u_y^e(x) dx + \overline{V} u_y^e(x)|_{\Gamma_V^e} + \overline{M} \theta_y^e(x)|_{\Gamma_M^e} \right\}$

C^1 continuity

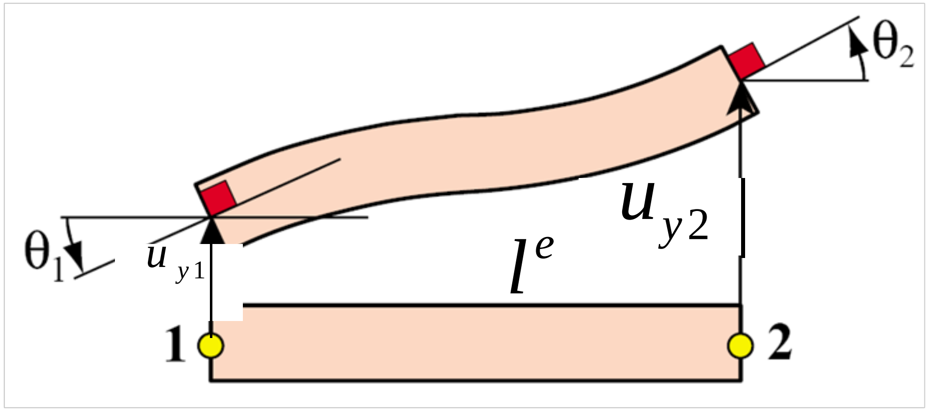
- Displacement u_y that is C^1 continuous: Both u_y and $\theta = {du_y}/{dx}$ are continuous.



- Displacement u_y that is C^0 continuous: u_y is continuous but $\theta = {du_y}/{dx}$ is discontinuous.



Two-noded beam element



- We need interpolation for both displacements and slopes at the ends of the beam (C^1 continuity)
- Our element degrees of freedom are taken as the following:

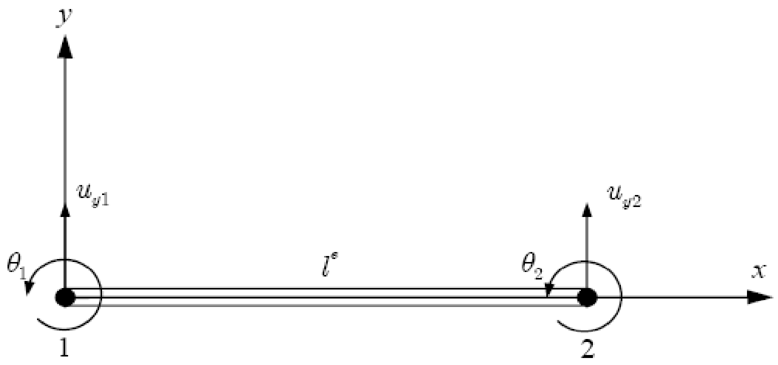
$$\{d^e\} = \begin{Bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{Bmatrix}$$

- The corresponding conjugate nodal forces are:

$$\{f^e\} = \begin{Bmatrix} F_{y1} \\ M_1 \\ F_{y2} \\ M_2 \end{Bmatrix}$$

Note that: $M_I \neq M(X_I),$ $I = 1, 2$

Two-noded beam element: Sign conventions

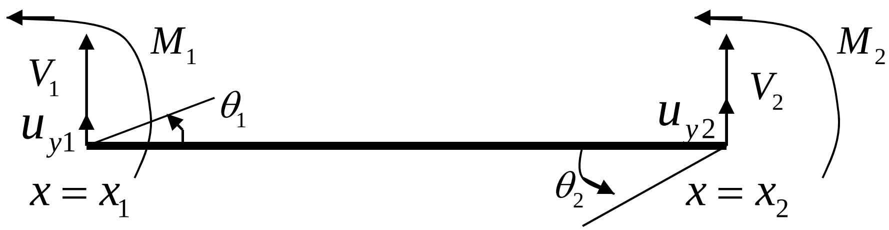


- Please note the convention for positive **nodal displacements and slopes** at both ends (**positive y-axis - upwards** and **counterclockwise**, respectively)

- As a result, the notation for the conjugate nodal forces is as follows:

F_{y1}, F_{y2} are positive when they point in the positive y-axis

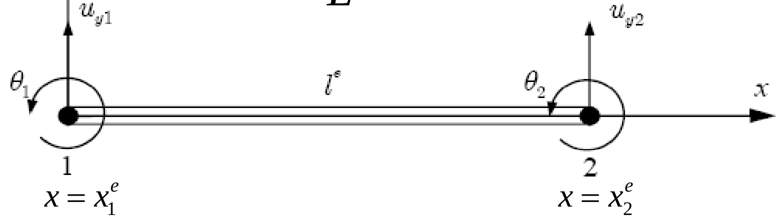
M_{y1}, M_{y2} are positive when they are anticlockwise



Interpolation functions for 2-noded beam element

$$x = \frac{L^e}{2}(1+\xi) + x_1^e, \quad -1 \le \xi \le 1$$

$$\xi = \frac{2(x^e - x_1^e)}{L^e} - 1, \quad -1 \le \xi \le 1$$



- We interpolate the deflection along the beam using the following Hermite interpolation functions for beam elements.

$$u_y^e(x) = [N_{u1} \quad N_{\theta 1} \quad N_{u2} \quad N_{\theta 2}] \begin{Bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{Bmatrix}$$

$$N_{u1} = \frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_{\theta 1} = \frac{L^e}{8}(1-\xi)^2(1+\xi)$$

$$N_{u2} = \frac{1}{4}(1+\xi)^2(2-\xi)$$

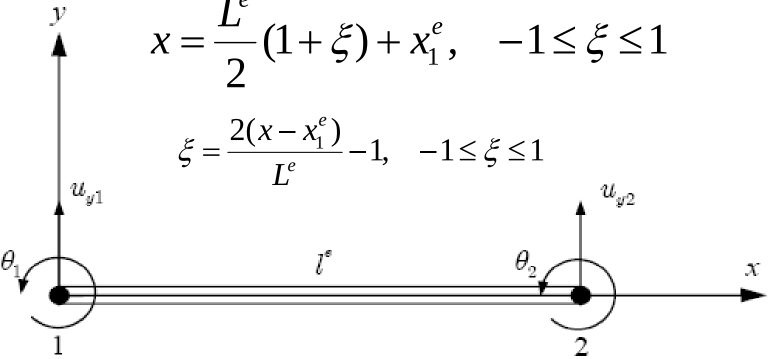
$$N_{\theta 2} = \frac{L^e}{8}(1+\xi)^2(\xi-1)$$

$$u_y^e(x) = \underbrace{[N^e]}_{\text{row vector}} \underbrace{\{d^e\}}_{\text{column vector}}$$

How the interpolation functions are computed?

$$x = \frac{L^e}{2}(1+\xi) + x_1^e, \quad -1 \le \xi \le 1$$

$$\xi = \frac{2(x - x_1^e)}{L^e} - 1, \quad -1 \le \xi \le 1$$



$x = x_1$, $\xi = -1$ (node 1); $x = x_2$, $\xi = +1$ (node 2)

$$Chain\ rule: \frac{du_y}{d\xi} = \frac{du_y}{dx}\frac{dx}{d\xi} = \frac{du_y}{dx}\frac{L^e}{2}$$

- We are looking for an approximation of the form:

$$u_y(x) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 = \begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\frac{du_y(x)}{d\xi} = a_1 + 2a_2\xi + 3a_3\xi^2$$

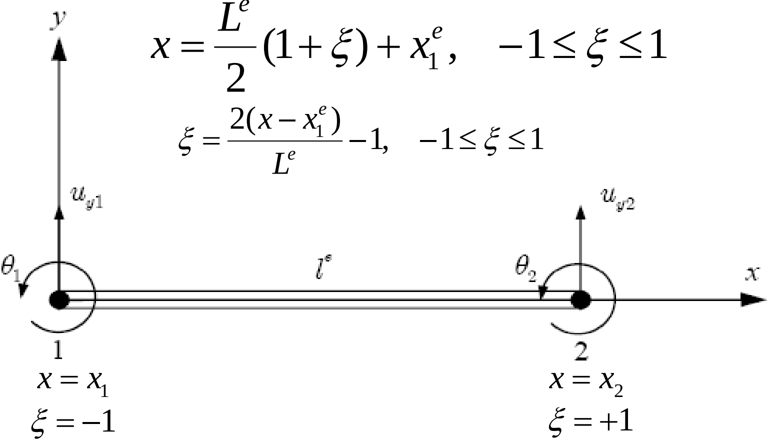
- We compute by interpolation using the values of deflection and slope at the ends of the element:

$$\left\{\begin{array}{l} \xi = -1:\ u_y|_{\xi=-1} = u_{y1},\ u_{y1} = a_0 - a_1 + a_2 - a_3 \\ \xi = -1: \frac{du_y}{d\xi}|_{\xi=-1} = \frac{du_y}{dx}|_{x=x_1}\frac{L^e}{2} = \theta_1\frac{L^e}{2}, \frac{L^e}{2}\theta_1 = a_1 - 2a_2 + 3a_3 \\ \xi = 1:\ u_y|_{\xi=1} = u_{y2}, u_{y2} = a_0 + a_1 + a_2 + a_3 \\ \xi = 1:\ \frac{du_y}{d\xi}|_{\xi=1} = \frac{du_y}{dx}|_{x=x_2}\frac{L^e}{2} = \theta_2\frac{L^e}{2}, \frac{L^e}{2}\theta_2 = a_1 + 2a_2 + 3a_3 \end{array}\right\} \Rightarrow \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1/2 & L^e/8 & 1/2 & -L^e/8 \\ -3/4 & -L^e/8 & 3/4 & -L^e/8 \\ 0 & -L^e/8 & 0 & L^e/8 \\ 1/4 & L^e/8 & -1/4 & L^e/8 \end{bmatrix}\begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix}$$

How the interpolation functions are computed?

$$x = \frac{L^e}{2}(1+\xi) + x_1^e, \quad -1 \le \xi \le 1$$

$$\xi = \frac{2(x - x_1^e)}{L^e} - 1, \quad -1 \le \xi \le 1$$



$$u_y(x) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 = \begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1/2 & L^e/8 & 1/2 & -L^e/8 \\ -3/4 & -L^e/8 & 3/4 & -L^e/8 \\ 0 & -L^e/8 & 0 & L^e/8 \\ 1/4 & L^e/8 & -1/4 & L^e/8 \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix}$$

- With substitution, we obtain:

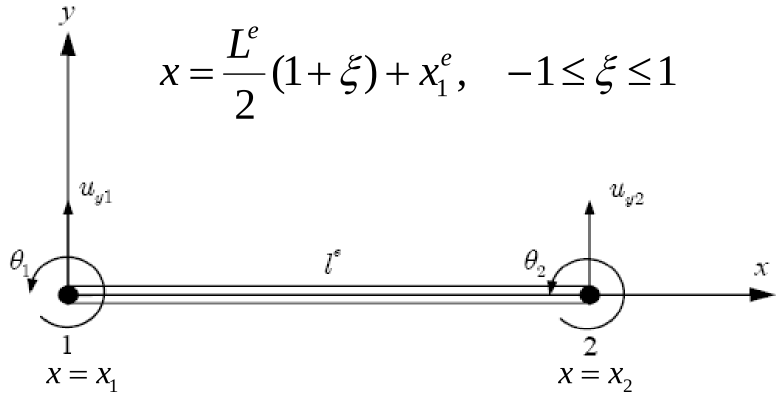
$$u_y(x) = \begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \end{bmatrix} \begin{bmatrix} 1/2 & L^e/8 & 1/2 & -L^e/8 \\ -3/4 & -L^e/8 & 3/4 & -L^e/8 \\ 0 & -L^e/8 & 0 & L^e/8 \\ 1/4 & L^e/8 & -1/4 & L^e/8 \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix}$$

- This can finally be simplified as:

$$u_y(x) = \begin{bmatrix} \underbrace{\frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3}_{N_{u1}} & \underbrace{\frac{L^e}{2}\left(\frac{1}{4} - \frac{1}{4}\xi - \frac{1}{4}\xi^2 + \frac{1}{4}\xi^3\right)}_{N_{\theta 1}} & \underbrace{\frac{1}{2} + \frac{3}{4}\xi - \frac{1}{4}\xi^3}_{N_{u2}} & \underbrace{\frac{L^e}{2}\left(-\frac{1}{4} - \frac{1}{4}\xi + \frac{1}{4}\xi^2 + \frac{1}{4}\xi^3\right)}_{N_{\theta 2}} \end{bmatrix} \begin{bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{bmatrix}$$

Interpolation functions for 2-noded beam element

$$x = \frac{L^e}{2}(1+\xi) + x_1^e, \quad -1 \le \xi \le 1$$



$$\xi = \frac{2(x - x_1^e)}{L^e} - 1, \quad -1 \le \xi \le 1$$

$$N_{u1} = \frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_{\theta 1} = \frac{L^e}{8}(1-\xi)^2(1+\xi)$$

$$N_{u2} = \frac{1}{4}(1+\xi)^2(2-\xi)$$

$$N_{\theta 2} = \frac{L^e}{8}(1+\xi)^2(\xi-1)$$

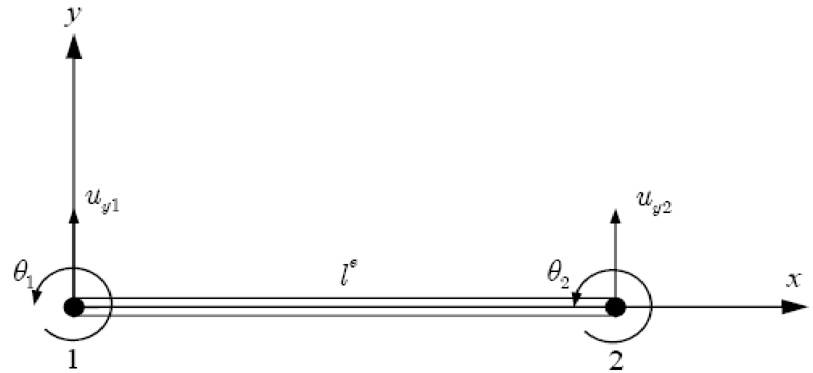
- Use change rule to find derivatives with respect to *x*, e.g.

$$\frac{dN_{u1}}{dx} = \frac{dN_{u1}}{d\xi}\frac{d\xi}{dx}$$

$$\frac{dN_{u1}}{d\xi} = -\frac{3}{4}(1-\xi^2)$$

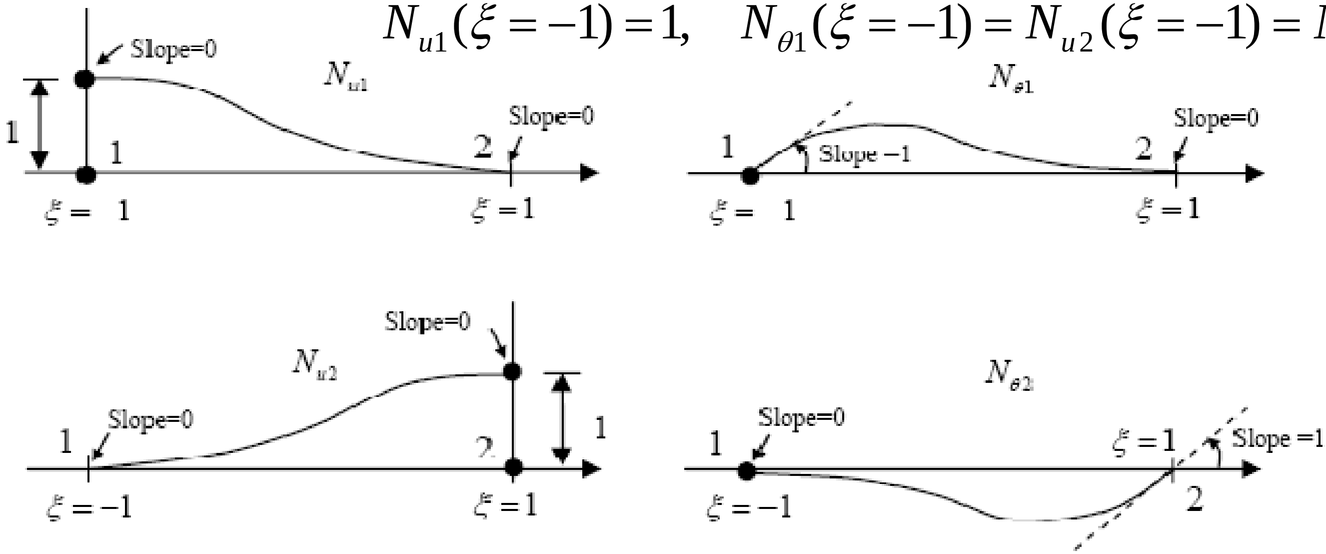
$$\frac{d\xi}{dx} = \frac{2}{L^e}$$

Continuity of displacements



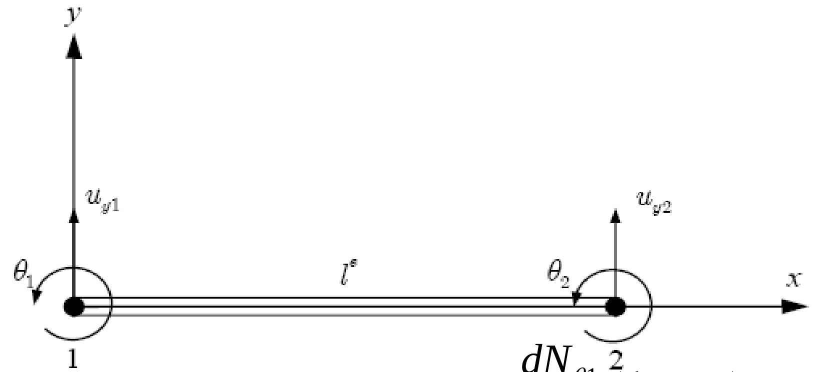
$$u_y^e(x) = [N_{u1} \quad N_{\theta 1} \quad N_{u2} \quad N_{\theta 2}] \begin{Bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{Bmatrix}$$

$\xi = -1$ $\quad N_{u1}(\xi = -1) = 1, \quad N_{\theta 1}(\xi = -1) = N_{u2}(\xi = -1) = N_{\theta 2}(\xi = -1) = 0$



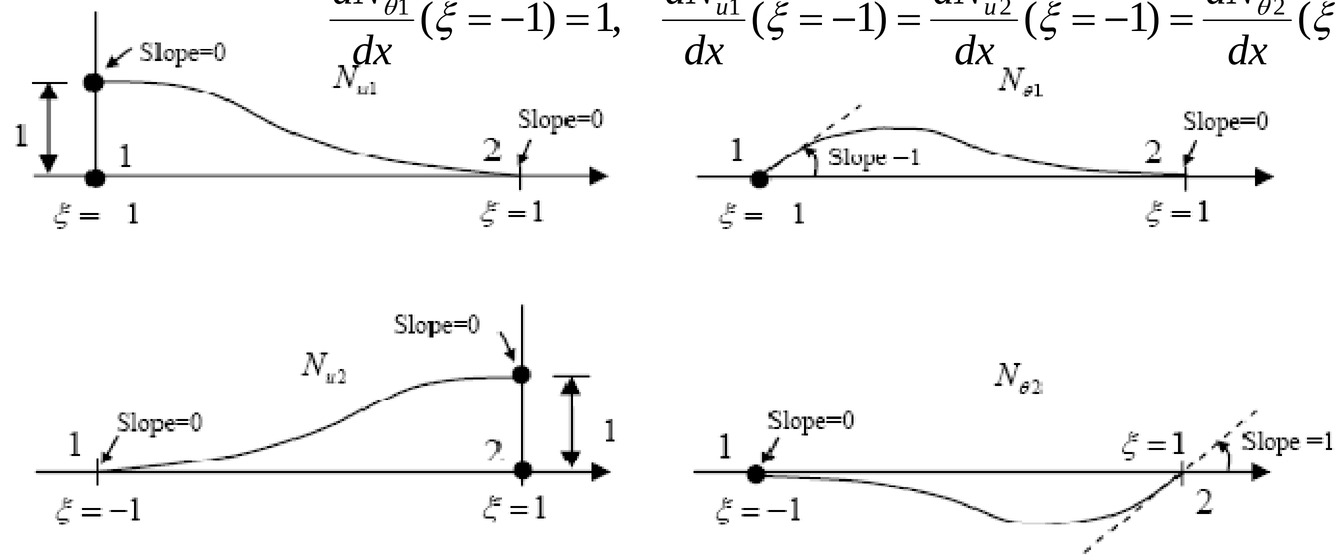
- Note that at ξ=-1 (left node), $u_y^e(left\ node) = u_{y1}$. Similarly at the right node.

Continuity of slopes



$$\theta(x) \equiv \frac{du_y^e(x)}{dx} = \left[\frac{dN_{u1}}{dx} \quad \frac{dN_{\theta 1}}{dx} \quad \frac{dN_{u2}}{dx} \quad \frac{dN_{\theta 2}}{dx}\right] \begin{Bmatrix} u_{y1} \\ \theta_1 \\ u_{y2} \\ \theta_2 \end{Bmatrix}$$

$$\frac{dN_{\theta 1}}{dx}(\xi=-1)=1, \quad \frac{dN_{u1}}{dx}(\xi=-1)=\frac{dN_{u2}}{dx}(\xi=-1)=\frac{dN_{\theta 2}}{dx}(\xi=-1)=0$$

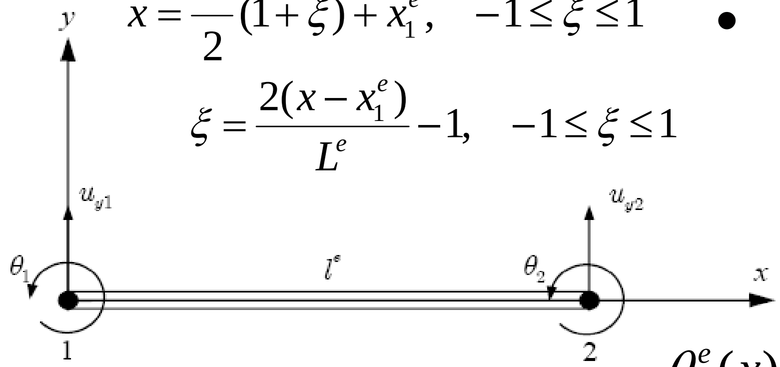


- Note that at ξ=-1 (left node), $\theta^e(left\ node) = \theta_1$. Similarly at the right node.

Curvature calculation

$$x = \frac{L^e}{2}(1+\xi) + x_1^e, \quad -1 \le \xi \le 1$$

$$\xi = \frac{2(x - x_1^e)}{L^e} - 1, \quad -1 \le \xi \le 1$$



$$N_{u1} = \frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_{\theta 1} = \frac{L^e}{8}(1-\xi)^2(1+\xi)$$

$$N_{u2} = \frac{1}{4}(1+\xi)^2(2-\xi)$$

$$N_{\theta 2} = \frac{L^e}{8}(1+\xi)^2(\xi-1)$$

- We will need to also compute the curvature $\kappa = \frac{d^2u_y^e}{dx^2}$ to compute the strain energy. Starting from:

$$\theta^e(x) = \frac{du_y}{dx} = [\frac{dN_{u1}}{dx} \quad \frac{dN_{\theta 1}}{dx} \quad \frac{dN_{u2}}{dx} \quad \frac{dN_{\theta 2}}{dx}]\{d^e\} \Rightarrow$$

$$\kappa = \frac{d^2u_y^e}{dx^2} = [\frac{d^2N_{u1}}{dx^2} \quad \frac{d^2N_{\theta 1}}{dx^2} \quad \frac{d^2N_{u2}}{dx^2} \quad \frac{d^2N_{\theta 2}}{dx^2}]\{d^e\} \Rightarrow$$

$$\frac{d^2u_y^e}{dx^2} = \frac{4}{L^{e^2}}[\frac{d^2N_{u1}}{d\xi^2} \quad \frac{d^2N_{\theta 1}}{d\xi^2} \quad \frac{d^2N_{u2}}{d\xi^2} \quad \frac{d^2N_{\theta 2}}{d\xi^2}]\{d^e\} \Rightarrow$$

$$\frac{d^2u_y^e}{dx^2} = \underbrace{\frac{1}{L^e}[\frac{6\xi}{L^e} \quad 3\xi - 1 \quad -\frac{6\xi}{L^e} \quad 3\xi + 1]}_{B^e}\{d^e\} \Rightarrow$$

$$\boxed{\frac{d^2u_y^e}{dx^2} = [B^e]\{d^e\}}$$

Strain energy calculation

- Recall that: $U^e = \int_{\Omega^e} \frac{E^e I^e}{2} (\frac{d^2 u_y^e}{dx^2})^2 dx$

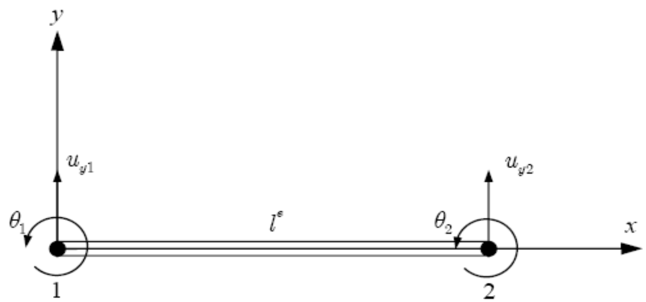
- Thus with our Hermite interpolation, $\frac{d^2 u_y^e}{dx^2} = [B^e]\{d^e\}$ we can write:

$$U^e = \frac{1}{2}\{d^e\}^T \underbrace{\int_{\Omega^e} [B^e]^T E^e I^e [B^e] dx^e}_{Beam\ element\ stiffness} \{d^e\}$$

- Recall that $\{d^e\} = [L^e]\{d\}$ (from global to local degrees of freedom), so the assembly of the strain energy term will give:

$$U = \frac{1}{2}\{d\}^T \sum_e ([L^e]^T \int_{-1}^{1} [B^e]^T E^e I^e [B^e] \underbrace{\underbrace{\frac{L^e}{2}}_{Jacobian} d\xi}_{dx} [L^e])\{d\}$$

Element stiffness



$$K^e = \int_{-1}^{1} B^{e^T} E^e I^e B^e \frac{L^e}{2} d\xi$$

- For constant $E^e I^e$ on the element, the stiffness $[K^e]$ is given as (use direct integration):

$$K^e = \frac{E^e I^e}{2L^{e^3}} \int_{-1}^{1} \begin{bmatrix} 36\xi^2 & 6\xi(3\xi-1)L^e & -36\xi^2 & 6\xi(3\xi+1)L^e \\ 6\xi(3\xi-1)L^e & (3\xi-1)^2 L^{e^2} & -6\xi(3\xi-1)L^e & (9\xi^2-1)L^{e^2} \\ -36\xi^2 & -6\xi(3\xi-1)L^e & 36\xi^2 & -6\xi(3\xi+1)L^e \\ 6\xi(3\xi+1)L & (9\xi^2-1)L^{e^2} & -6\xi(3\xi+1)L^e & (3\xi+1)^2 L^{e^2} \end{bmatrix} d\xi = \frac{E^e I^e}{L^{e^3}} \begin{bmatrix} 12 & 6L^e & -12 & 6L^e \\ 6L^e & 4L^{e^2} & -6L^e & 2L^{e^2} \\ -12 & -6L^e & 12 & -6L^e \\ 6L^e & 2L^{e^2} & -6L^e & 4L^{e^2} \end{bmatrix}$$

External work calculation

- The external work is: $W = \sum_e \left\{ \int_{\Omega^e} q(x) u_y^e(x) dx + V_1^e u_y^e(x_1^e) + V_2^e u_y^e(x_1^e) + M_1^e \theta_y^e(x_1^e) + M_2^e u_y^e(x_2^e) \right\}$
- Upon assembly all terms with *V* and *M* cancel at each node unless external force and/or moments are applied there.
- We can thus write:

$$W = \sum_e \left\{ \int_{\Omega^e} q(x) u_y^e(x) dx + \overline{V} u_y^e(x)|_{\Gamma_V^e} + \overline{M} \theta_y^e(x)|_{\Gamma_M^e} \right\}$$

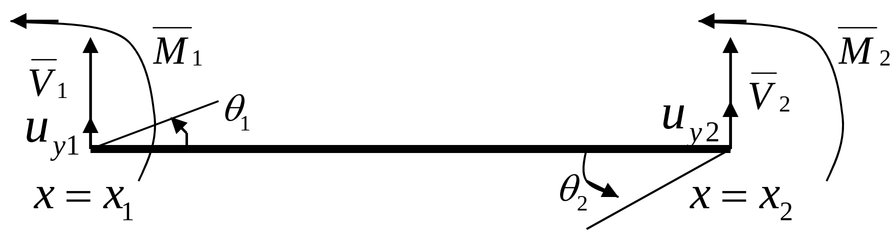
- The boundary Γ_V^e is non-zero only if the element e has at one of its ends a prescribed shear force $\overline{V}$.
- Similarly, the boundary Γ_M^e is non-zero only if the element e has at one of its ends a prescribed moment $\overline{M}$.
- Note that the sign notation for $\overline{V}$ and $\overline{M}$ is consistent with the sign convention for V_1^e, V_2^e and M_1^e, M_2^e , respectively.

External work calculation

- Recall that: $$W=\sum_{e}\left\{\int_{\Omega^e} q(x)u_y^e(x)dx+\overline{V}u_y^e(x)|_{\Gamma_V^e}\ +\overline{M}\theta_y^e(x)|_{\Gamma_M^e}\right\}$$

- For example, if we only account for prescribed moments and shear force at both ends, the above equation is written explicitly as:

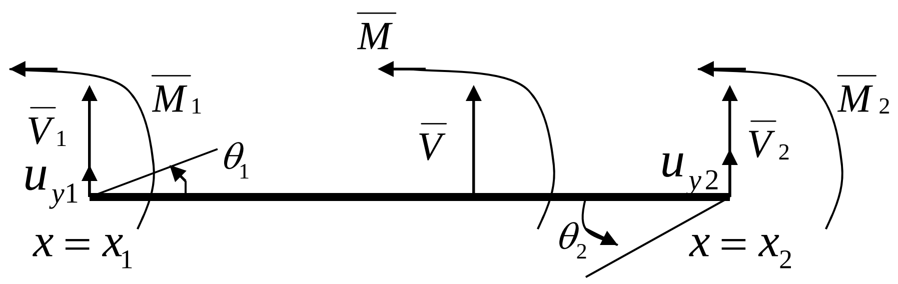
$$W=\sum_{e}\left\{\int_{\Omega^e} q(x)u_y^e(x)dx\right\}+\overline{V}_2u_2+\overline{V}_1u_1+\overline{M}_2\theta_2+\overline{M}_1\theta_1$$



External work calculation

- We will keep the general notation here as it will also allow us to account for shear force and moments applied even inside the element!

$$W=\sum_e\left\{\int_{\Omega^e} q(x)u_y^e(x)dx+\overline{V}u_y^e(x)\big|_{\Gamma_V^e}+\overline{M}\theta_y^e(x)\big|_{\Gamma_M^e}\right\}$$



- We will consider these cases in our following derivations – but please note that if you apply a moment or shear force at a point, you will be better served to make that point a finite element node!!

External work calculation

- Recall that: $W = \sum_e \left\{ \int_{\Omega^e} q(x) u_y^e(x) dx + \overline{V} u_y^e(x) |_{\Gamma_V^e} + \overline{M} \theta_y^e(x) |_{\Gamma_M^e} \right\}$
- We assume known distributed load, applied concentrated load and applied concentrated moments. Applying the Hermite interpolation:

$$W^e = \{d^e\}^T \left\{ \int_{\Omega^e} q(x) [N^e]^T dx + [N^e]^T \overline{V} |_{\Gamma_V^e} + \frac{d[N^e]^T}{dx} \overline{M} |_{\Gamma_M^e} \right\}$$

- Note that in this expression, only the elements *e* that have boundaries (i.e. one of their end points) on the boundaries Γ_V, Γ_M contribute to the last two terms.

External work calculation

$$W^e = \{d^e\}^T \left\{ \int_{\Omega^e} q(x)[N^e]^T dx + [N^e]^T \overline{V} \Big|_{\Gamma_V^e} + \frac{d[N^e]^T}{dx} \overline{M} \Big|_{\Gamma_M^e} \right\}$$

- Also note that with our notation, $[N^e]$ and $[B^e]$ are row vectors and $\{d^e\}$ is a column vector. Assembling (transform from local to global degrees of freedom) then gives:

$$W = \{d\}^T \sum_e [L^e]^T \left\{ \int_{\Omega^e} q(x)[N^e]^T dx + [N^e]^T \overline{V} \Big|_{\Gamma_V^e} + \frac{d[N^e]^T}{dx} \overline{M} \Big|_{\Gamma_M^e} \right\}$$

$$\equiv \{d\}^T \sum_e [L^e]^T \left\{ \int_{\Omega^e} q(x)[N^e]^T dx + \underbrace{[N^e]^T}_{\substack{column \\ vector}} \overline{V} \Big|_{\Gamma_V^e} + \underbrace{\frac{d[N^e]^T}{dx}}_{\substack{column \\ vector}} \overline{M} \Big|_{\Gamma_M^e} \right\}$$

Minimization of total potential energy

$$\min_{\{d^e\}} \Pi = \min_{\{d\}} \frac{1}{2}\{d\}^T \sum_e ([L^e]^T \int_{\Omega^e} [B^e]^T E^e I^e [B^e] dx^e [L^e])\{d\} -$$

$$\{d\}^T \sum_e [L^e]^T \left\{ \int_{\Omega^e} q(x)[N^e]^T dx + [N^e]^T \overline{V}\,|_{\Gamma_V^e} + \frac{d[N^e]^T}{dx}\overline{M}\,|_{\Gamma_M^e} \right\}$$

- This minimization problem results in:

$$\underbrace{\sum_e ([L^e]^T \int_{\Omega^e} [B^e]^T E^e I^e [B^e] dx^e [L^e])}_{Global\ stiffness\,[K]}\{d\} = \underbrace{\sum_e [L^e]^T \left\{ \int_{\Omega^e} q(x)[N^e]^T dx + [N^e]^T \overline{V}\,|_{\Gamma_V^e} + \frac{d[N^e]^T}{dx}\overline{M}\,|_{\Gamma_M^e} \right\}}_{Global\ force\ vector\,\{F\}}$$

Final finite element equations

$$\sum_e \left([L^e]^T \int_{\Omega^e} [B^e]^T E^e I^e [B^e] dx^e [L^e] \right) \{d\} = \sum_e [L^e]^T \left\{ \int_{\Omega^e} q(x)[N^e]^T dx + [N^e]^T \overline{V} \Big|_{\Gamma_V^e} + \frac{d[N^e]^T}{dx} \overline{M} \Big|_{\Gamma_M^e} \right\}$$

- Note that the above minimization process is with respect to the nodal displacements $\{d_F\}$ (i.e. excluding the DOF with prescribed displacement or rotation).

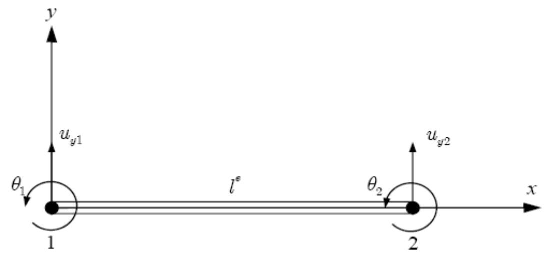
- As was done in earlier lectures, by splitting the stiffness matrix, we finally obtain:

$$[K_F]\{d_F\} = f_{\Omega F} + f_{\Gamma F} - [K_{EF}]^T \{\overline{d}_E\}$$

Distributed loads ($f_{\Omega F}$)

Boundary Loads (forces,moments) ($f_{\Gamma F}$)

Uniform distributed load

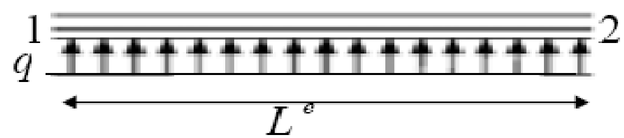


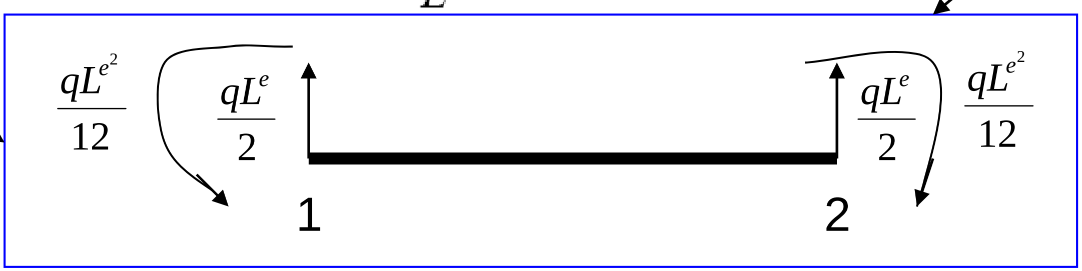
$$f_\Omega^e = \int_{\Omega^e} q(x)[N^e]^T dx$$

- For uniform over the element load q the first term gives:

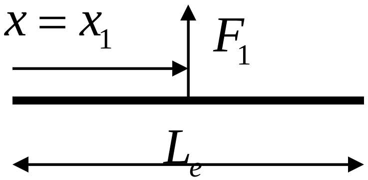
$$f_\Omega^e = \int_{\Omega^e} q(x)[N^e]^T dx = \frac{L^e}{2}\int_{-1}^{1} q(x)\begin{Bmatrix} N_{u1} \\ N_{\vartheta 1} \\ N_{u2} \\ N_{\vartheta 2} \end{Bmatrix} d\xi \underbrace{\Rightarrow}_{\substack{perform \\ integration \\ for \\ constant\ q}}$$

$$f^e = \frac{qL^e}{2}\begin{Bmatrix} 1 \\ \frac{L^e}{6} \\ 1 \\ -\frac{L^e}{6} \end{Bmatrix}$$





Concentrated load inside the element



$$f_\Omega^e = \int_{\Omega^e} q(x)[N^e]^T \, dx$$

- Assume a contentrated load at $x = x_1$. We write this load as a distributed load using a delta function (nice trick!):

$$q(x) = F_1 \delta(x - x_1)$$

- Substitution into the first term of the formula for f_Ω^e gives:

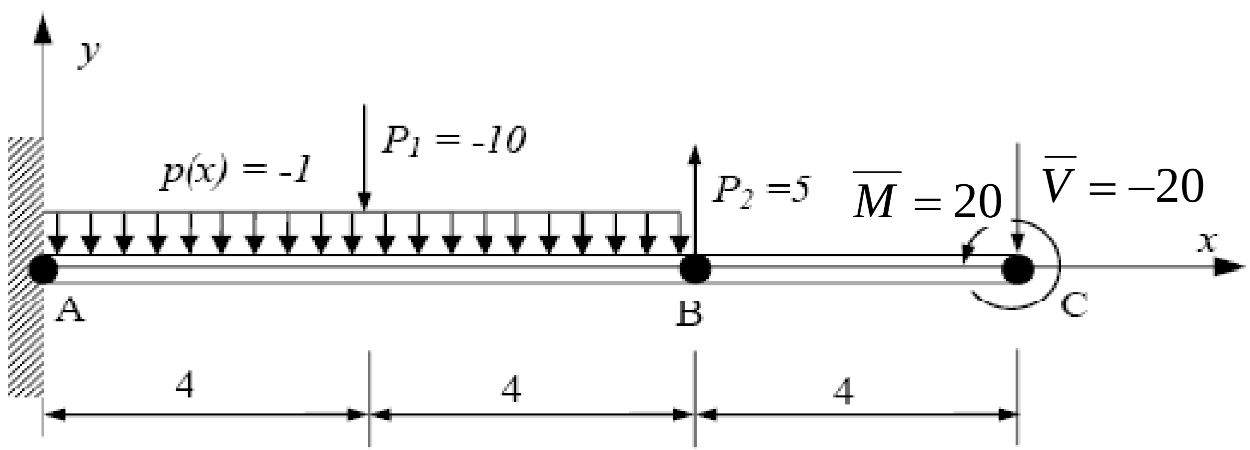
$$f_\Omega^e = \int_{\Omega^e} F_1 \delta(x - x_1)[N^e]^T \, dx \Rightarrow \quad f_\Gamma^e = F_1 [N^e(x_1)]^T$$

If the applied load is at a node e.g. x_2^e , then:

$$f_\Gamma^e = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} F_1$$

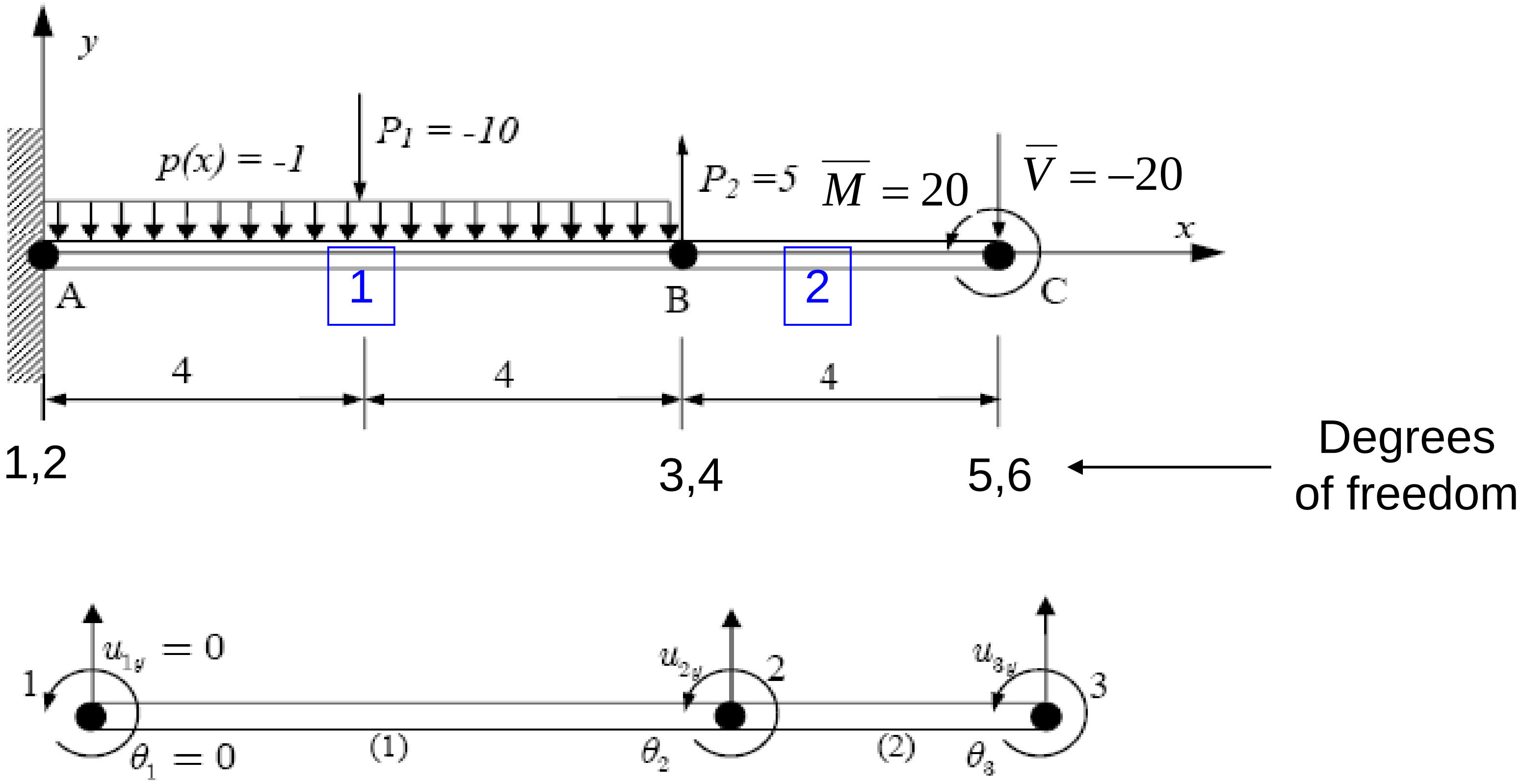
Example problem

- The beam ABC is clamped at the left side and simply supported at the right side. Dimensions are in m, forces in N and loading *q* in N/m . Also, *EI* = 10^4 Nm^2. At *x* =12m, $\overline{V} = -20Nt$ and $\overline{M} = 20N.m$. Find the deflection, shear forces and moments.

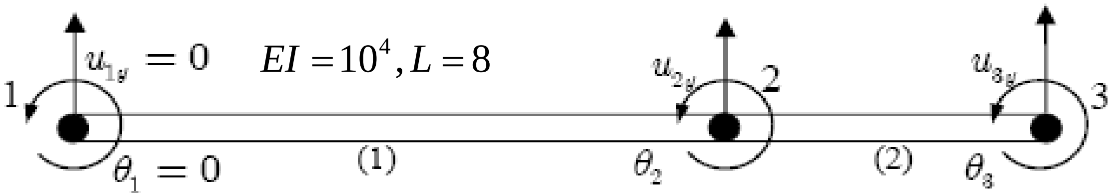


Finite element discretization

- We consider 2 beam elements as follows:



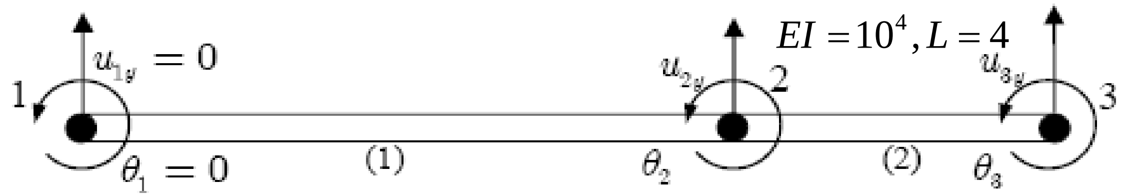
Stiffness of element 1



$$K^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} = 10^3 \begin{bmatrix} 0.23 & 0.94 & -0.23 & 0.94 \\ 0.94 & 5.00 & -0.94 & 2.50 \\ -0.23 & -0.94 & 0.23 & -0.94 \\ 0.94 & 2.50 & -0.94 & 5.00 \end{bmatrix}$$

(Columns labelled 1, 2, 3, 4; rows labelled 1, 2, 3, 4.)

Stiffness of element 2



$$K^e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} = 10^3 \begin{bmatrix} 1.88 & 3.75 & -1.88 & 3.75 \\ 3.75 & 10.00 & -3.75 & 5.00 \\ -1.88 & -3.75 & 1.88 & -3.75 \\ 3.75 & 5.00 & -3.75 & 10.00 \end{bmatrix}$$

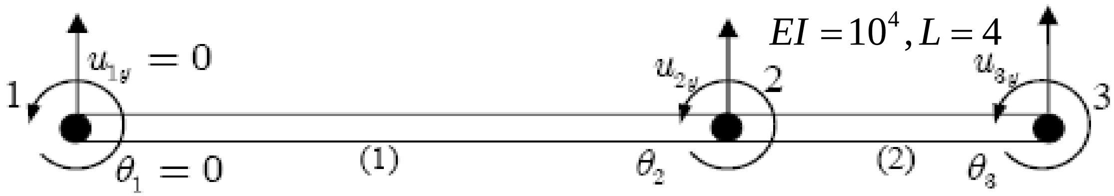
(Columns and rows of the numerical matrix are labelled 3, 4, 5, 6.)

Global stiffness

$$K = 10^3 \begin{bmatrix} 0.23 & 0.94 & -0.23 & 0.94 & 0 & 0 \\ 0.94 & 5.00 & -0.94 & 2.50 & 0 & 0 \\ -0.23 & -0.94 & 2.11 & 2.81 & -1.88 & 3.75 \\ 0.94 & 2.50 & 2.81 & 15.00 & -3.75 & 5.00 \\ 0 & 0 & -1.88 & -3.75 & 1.88 & -3.75 \\ 0 & 0 & 3.75 & 5.00 & -3.75 & 10.00 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

(column indices: 1 2 3 4 5 6)

Boundary moments and forces



$$f_\Gamma^e = [N^e]^T \overline{V} \,|_{\Gamma_V^e} + \frac{d[N^e]^T}{dx} \overline{M} \,|_{\Gamma_M^e}$$

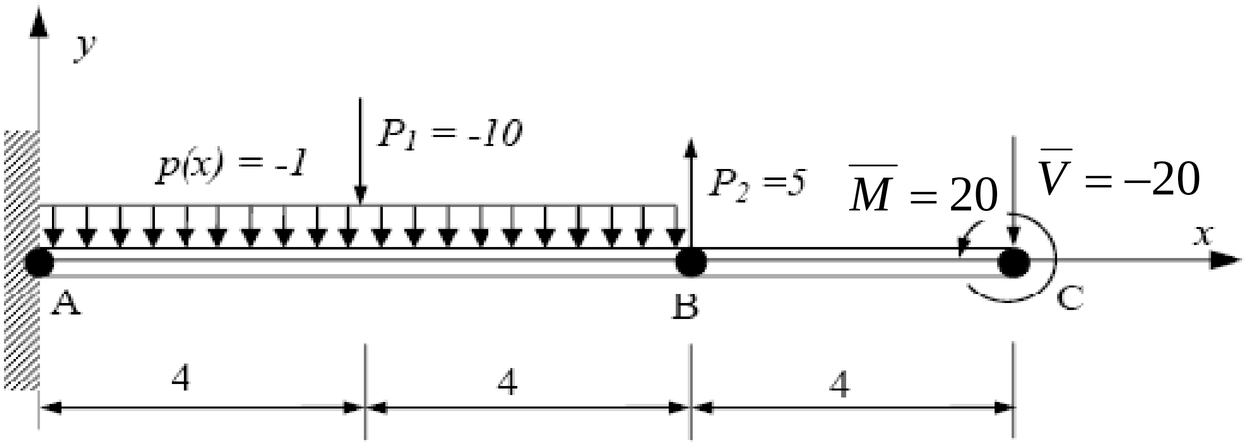
- Element 1 has no boundary with applied *V* or *M*.
- Element 2 has $\overline{V} = -20Nt$ and $\overline{M} = 20N.m$ at its right end.

$$f_\Gamma^2 = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \overline{V} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \overline{M} = \begin{Bmatrix} 0 \\ 0 \\ -20N \\ 20N \end{Bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

- Assembly of this vector gives:

$$f_\Gamma = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -20N \\ 20N \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Distributed load and concentrated load inside the element



$$N_{u1} = \frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_{\theta 1} = \frac{L^e}{8}(1-\xi)^2(1+\xi)$$

$$N_{u2} = \frac{1}{4}(1+\xi)^2(2-\xi)$$

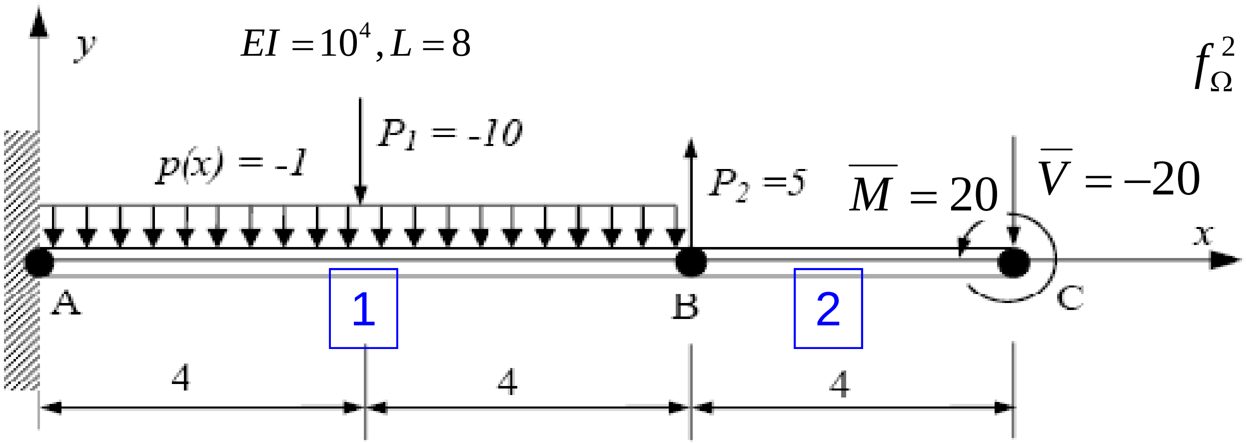
$$N_{\theta 2} = \frac{L^e}{8}(1+\xi)^2(\xi-1)$$

- Element 1 has distributed load q=-1 and concentrated load P_1=-10Nt.

$\xi = 0$

$$f_\Omega^1 = \frac{qL}{2}\begin{Bmatrix} 1 \\ \frac{L}{6} \\ 1 \\ -\frac{L}{6} \end{Bmatrix} + \underbrace{N^{1^T}(\xi=0)P_A}_{[N^1(\xi)]^T P_1|_{\xi=0}} = \frac{(-1)8}{2}\begin{Bmatrix} 1 \\ \frac{8}{6} \\ 1 \\ -\frac{8}{6} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{Bmatrix}(-10) = \begin{Bmatrix} -9 \\ -15.3 \\ -9 \\ 15.3 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Distributed load

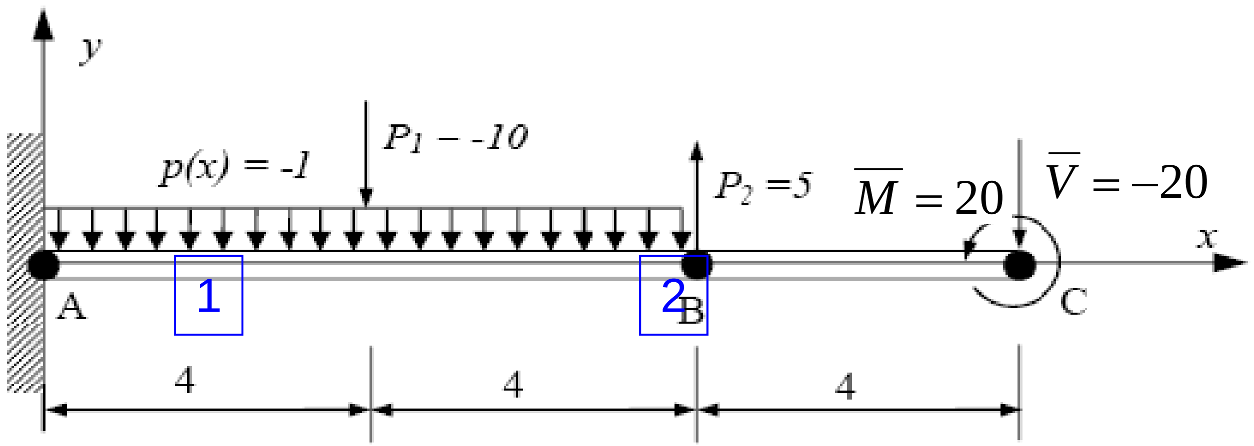


$$f_\Omega^2 = [N^2(\xi)]^T P_2 |_{\xi=-1}$$

- Element 2 has concentrated load P_2 at an end point $\xi = -1$

$$f_\Omega^2 = N^{2^T}(\xi = -1) P_2 = \begin{Bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

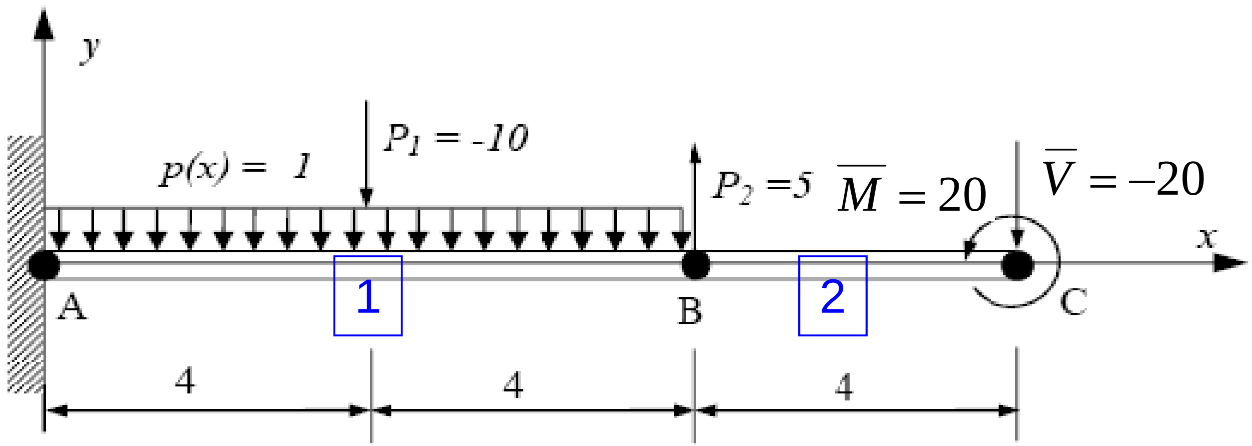
Assembled load



- Assembling all load contributions gives:

$$f_\Omega^1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -20N \\ 20N \end{Bmatrix} + \begin{Bmatrix} -9 \\ -15.3 \\ -9 \\ 15.3 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -9 \\ -15.3 \\ -4 \\ 15.3 \\ -20 \\ 20 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

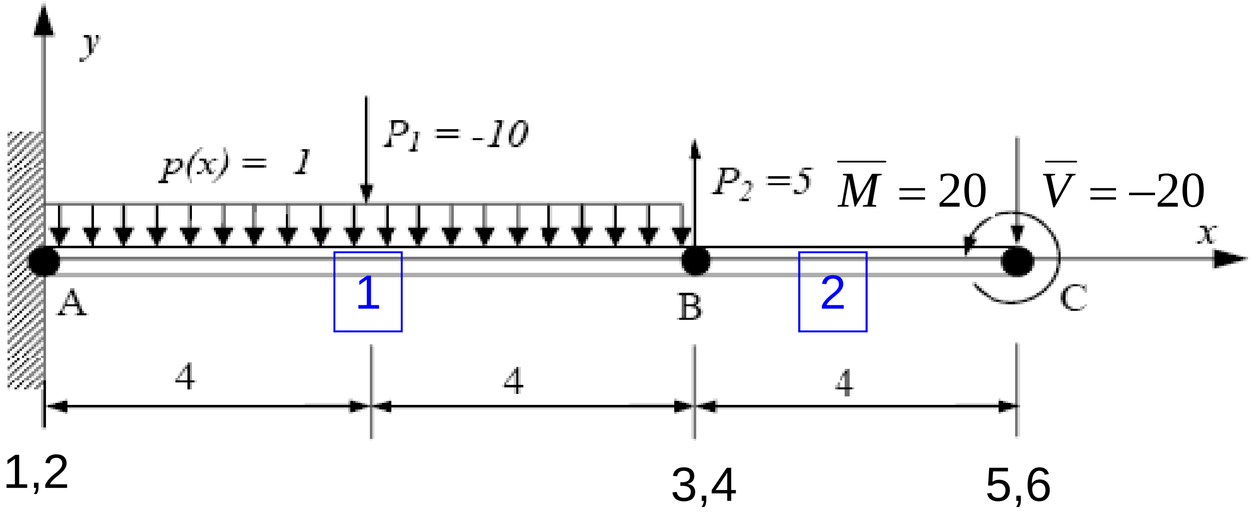
Solution step



$$10^3\begin{bmatrix} 0.23 & 0.94 & -0.23 & 0.94 & 0 & 0 \\ 0.94 & 5.00 & -0.94 & 2.50 & 0 & 0 \\ -0.23 & -0.94 & 2.11 & 2.81 & -1.88 & 3.75 \\ 0.94 & 2.50 & 2.81 & 15.00 & -3.75 & 5.00 \\ 0 & 0 & -1.88 & -3.75 & 1.88 & -3.75 \\ 0 & 0 & 3.75 & 5.00 & -3.75 & 10.00 \end{bmatrix} \begin{Bmatrix} u_{y1}=0 \\ \theta_{y1}=0 \\ u_{y2} \\ \theta_{y2} \\ u_{y3} \\ \theta_{y3} \end{Bmatrix} = \begin{Bmatrix} -9+R_{u1} \\ -15.3+R_{\vartheta 1} \\ -4 \\ 15.3 \\ -20 \\ 20 \end{Bmatrix}$$

- Note that we applied the essential boundary conditions and included the corresponding reaction force and moment at this location

Solution step



$$10^3 \begin{bmatrix} 2.11 & 2.81 & -1.88 & 3.75 \\ 2.81 & 15.00 & -3.75 & 5.00 \\ -1.88 & -3.75 & 1.88 & -3.75 \\ 3.75 & 5.00 & -3.75 & 10.00 \end{bmatrix} \begin{Bmatrix} u_{y2} \\ \theta_{y2} \\ u_{y3} \\ \theta_{y3} \end{Bmatrix} = \begin{Bmatrix} -4 \\ 15.3 \\ -20 \\ 20 \end{Bmatrix} \Rightarrow \begin{Bmatrix} u_{y2} \\ \theta_{y2} \\ u_{y3} \\ \theta_{y3} \end{Bmatrix} = \begin{Bmatrix} -0.55 \\ -0.11 \\ -1.03 \\ -0.12 \end{Bmatrix}$$

$$10^3 \begin{bmatrix} -0.23 & 0.94 & 0 & 0 \\ -0.94 & 2.50 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{y2} \\ \theta_{y2} \\ u_{y3} \\ \theta_{y3} \end{Bmatrix} = \begin{Bmatrix} -9 + R_{u1} \\ -15.3 + R_{\vartheta 1} \end{Bmatrix} \Rightarrow \begin{Bmatrix} R_{u1} \\ R_{\vartheta 1} \end{Bmatrix} = \begin{Bmatrix} 33N \\ 252N.m \end{Bmatrix}$$

Displacement field

- For element 1:

$$u_y^1(x) = [N_{u1}^1 \quad N_{\theta 1}^1 \quad N_{u2}^1 \quad N_{\theta 2}^1]\begin{Bmatrix} 0 \\ 0 \\ u_{y2} \\ \theta_{y2} \end{Bmatrix} = N_{u2}^1 u_{y2} + N_{\theta 2}^1 \theta_{y2}$$

- For element 2:

$$u_y^2(x) = [N_{u1}^2 \quad N_{\theta 1}^2 \quad N_{u2}^2 \quad N_{\theta 2}^2]\begin{Bmatrix} u_{y2} \\ \theta_{y2} \\ u_{y3} \\ \theta_{y3} \end{Bmatrix} = N_{u1}^2 u_{y2} + N_{\theta 1}^2 \theta_{y2} + N_{u2}^2 u_{y3} + N_{\theta 2}^3 \theta_{y3}$$

Moments and shear forces

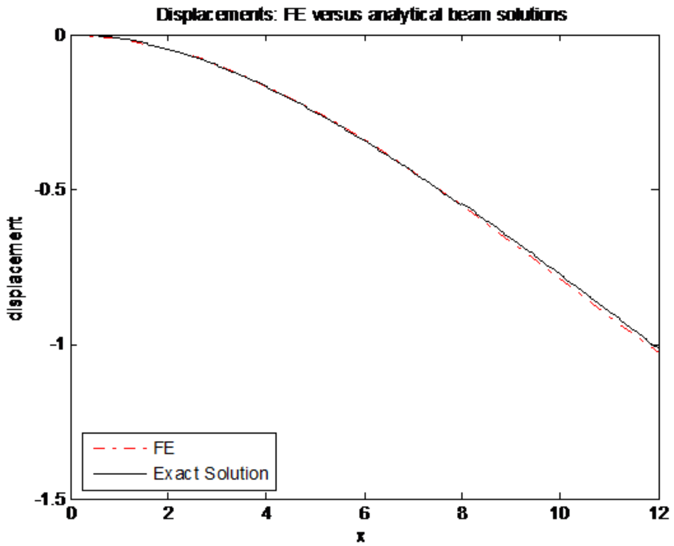
$$M^1 = EI\frac{d^2u_y^1}{dx^2} = EI[\frac{d^2N_{u1}^1}{dx^2} \quad \frac{d^2N_{\theta1}^1}{dx^2} \quad \frac{d^2N_{u2}^1}{dx^2} \quad \frac{d^2N_{\theta2}^1}{dx^2}]\{d^1\} = -236.67 + 23.76x$$

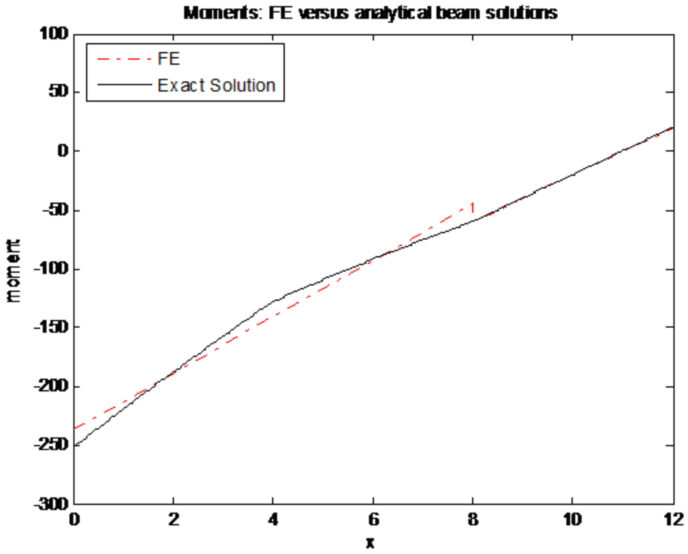
$$V^1 = -EI\frac{d^3u_y^1}{dx^2} = -EI[\frac{d^3N_{u1}^1}{dx^3} \quad \frac{d^3N_{\theta1}^1}{dx^3} \quad \frac{d^3N_{u2}^1}{dx^3} \quad \frac{d^3N_{\theta2}^1}{dx^3}]\{d^1\} = -23.76$$

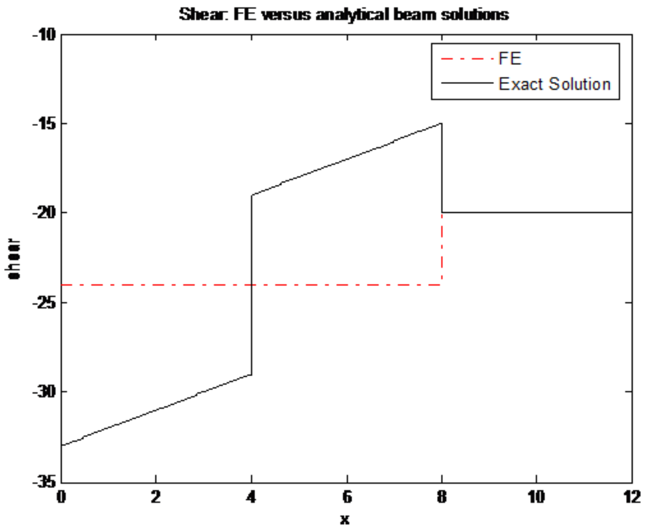
$$M^2 = EI\frac{d^2u_y^2}{dx^2} = EI[\frac{d^2N_{u1}^2}{dx^2} \quad \frac{d^2N_{\theta1}^2}{dx^2} \quad \frac{d^2N_{u2}^2}{dx^2} \quad \frac{d^2N_{\theta2}^2}{dx^2}]\{d^2\} = -222 + 20.25x$$

$$V^2 = -EI\frac{d^3u_y^2}{dx^2} = -EI[\frac{d^3N_{u1}^2}{dx^3} \quad \frac{d^3N_{\theta1}^2}{dx^3} \quad \frac{d^3N_{u2}^2}{dx^3} \quad \frac{d^3N_{\theta2}^2}{dx^3}]\{d^2\} = -20.25$$

Place nodes on location of concentrated load

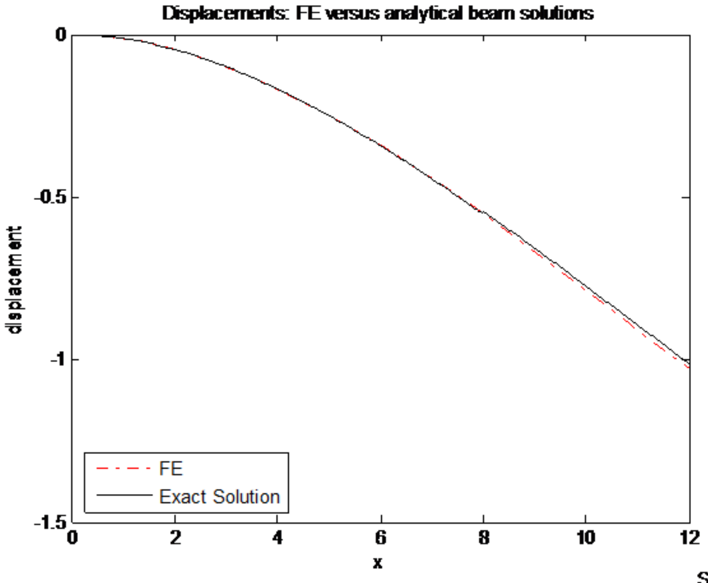


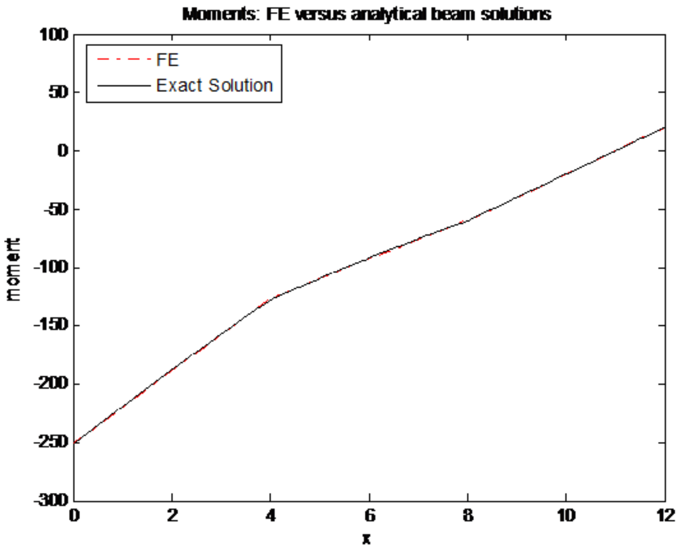


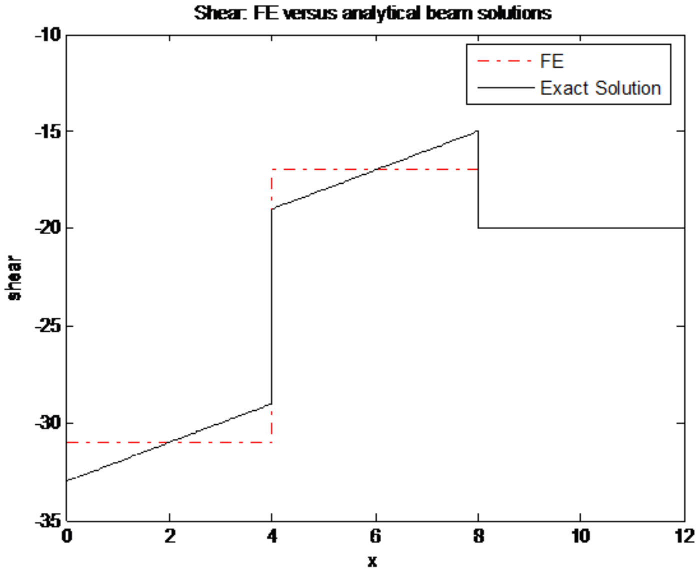


- To compute an accurate shear force, you need to split element 1 in more elements. For sure place a node at the application of load P_1.

Results with refined grids: 3 elements







Results with refined grids: 101 elements