MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

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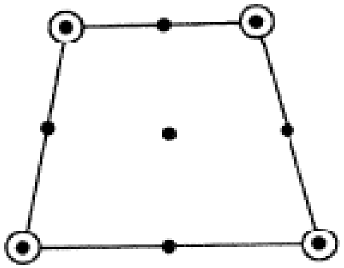
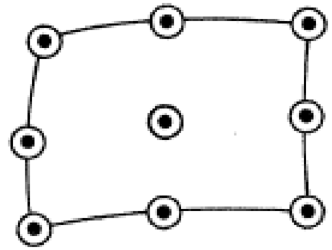
Isoparametric finite elements

- The basis functions used in the definition of the mapping T_e, do not have to be the same as those used for the approximation of functions.
- Let M be the number of basis functions used to define T_e and let N_e be the number of basis functions (nodes) used in the approximation of functions.
- Polynomials used to define geometry can be of higher order ($M>N_e$), equal ($M=N_e$) or lower ($M<N_e$) than those used for the approximation of the main fields. This defines super-parametric, isoparametric and sub-parametric finite elements, respectively.

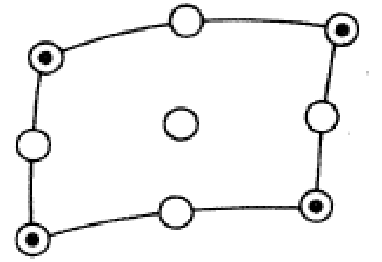
Sub-, iso- and super-parametric finite elements



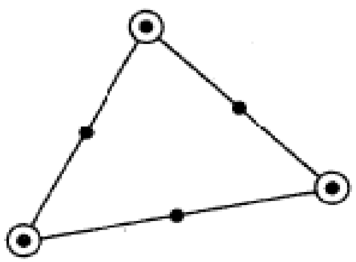
$M = 4$
$N_e = 9$



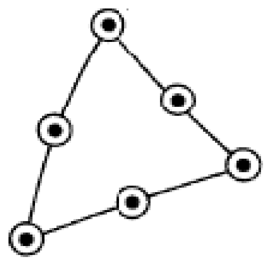
$M = 9$
$N_e = 9$



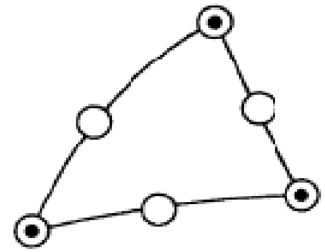
$M = 9$
$N_e = 4$



$M = 3$
$N_e = 6$



$M = 6$
$N_e = 6$



$M = 6$
$N_e = 3$

Quadrilateral elements: Bi-linear

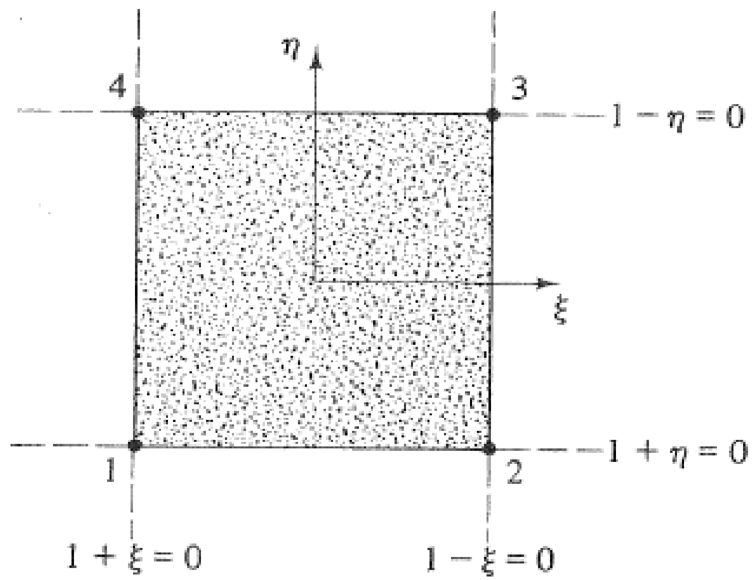
- We use tensor product of polynomials as discussed in 1D (Lagrange family)

$$\widehat{N}_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$$

$$\widehat{N}_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$$

$$\widehat{N}_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$$

$$\widehat{N}_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$



Quadrilateral elements: Bi-quadratic

- We use tensor product of polynomials as discussed in 1D (Lagrange family)

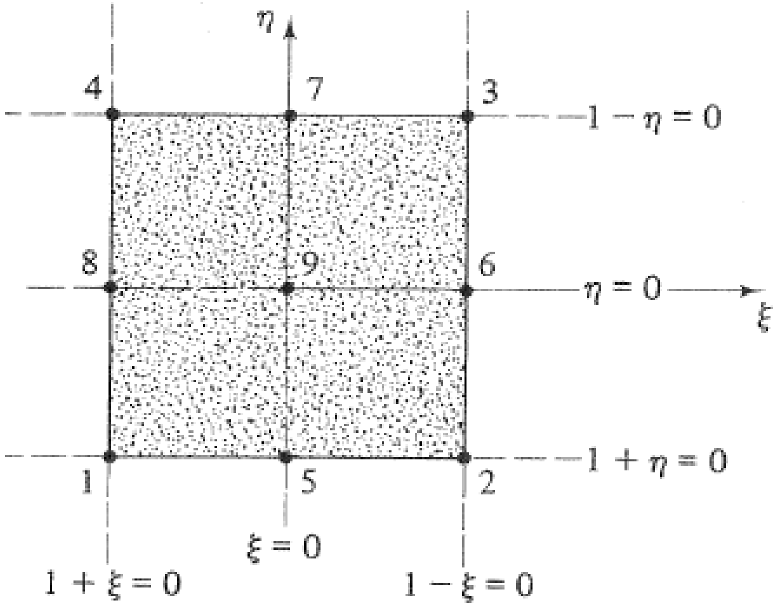
$$\widehat{N}_1(\xi,\eta)=\frac{1}{4}(\xi^2-\xi)(\eta^2-\eta),\ \widehat{N}_5(\xi,\eta)=\frac{1}{2}(1-\xi^2)(\eta^2-\eta)$$

$$\widehat{N}_2(\xi,\eta)=\frac{1}{4}(\xi^2+\xi)(\eta^2-\eta),\ \widehat{N}_6(\xi,\eta)=\frac{1}{2}(\xi^2+\xi)(1-\eta^2)$$

$$\widehat{N}_3(\xi,\eta)=\frac{1}{4}(\xi^2+\xi)(\eta^2+\eta),\ \widehat{N}_7(\xi,\eta)=\frac{1}{2}(1-\xi^2)(\eta^2+\eta)$$

$$\widehat{N}_4(\xi,\eta)=\frac{1}{4}(\xi^2-\xi)(\eta^2+\eta),\ \widehat{N}_8(\xi,\eta)=\frac{1}{2}(\xi^2-\xi)(1-\eta^2)$$

$$\widehat{N}_9(\xi,\eta)=(1-\xi^2)(1-\eta^2)$$

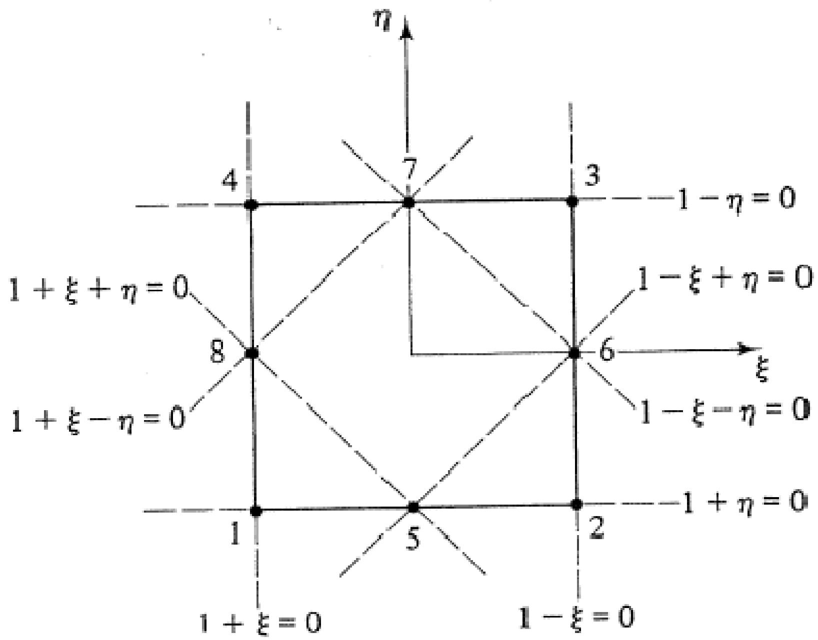


- Note that here we have one internal node (9).

Quadratic eight node element

- These type of elements are not derived from tensor product of 1D polynomials. They are called serendipity elements.
- To derive the shape function for node 1, we need a polynomial that vanishes on the following lines:

$$1-\xi, 1-\eta, 1+\xi+\eta$$



$$\widehat{N}_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)(-1-\xi-\eta),\ \widehat{N}_5(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1-\eta)$$

$$\widehat{N}_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)(-1+\xi-\eta),\ \widehat{N}_6(\xi,\eta)=\frac{1}{2}(1+\xi)(1-\eta^2)$$

$$\widehat{N}_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)(-1+\xi+\eta),\ \widehat{N}_7(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta)$$

$$\widehat{N}_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)(-1-\xi+\eta),\ \widehat{N}_8(\xi,\eta)=\frac{1}{2}(1-\xi)(1-\eta^2)$$

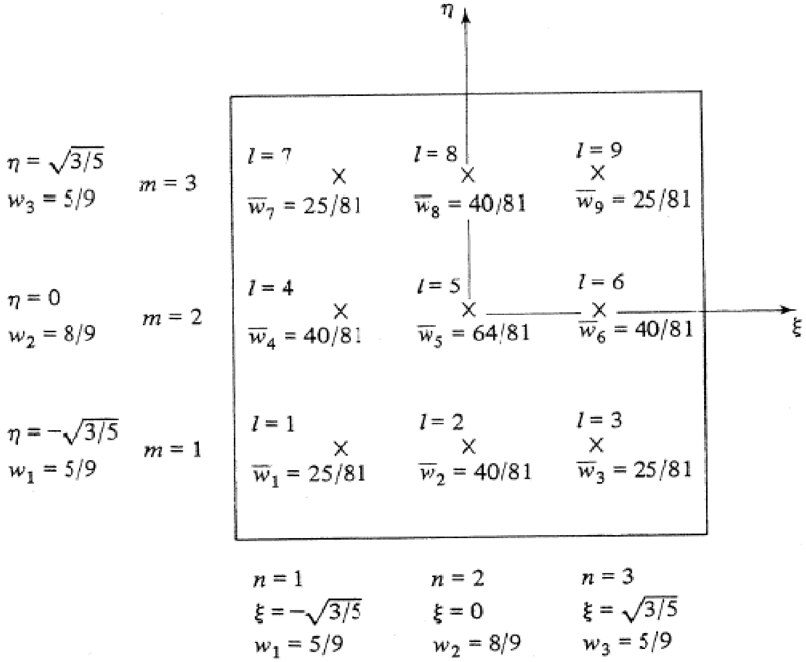
Quadrature rules

- Quadrature rules are defined from the 1D Gauss rules presented earlier as follows:

$$\int_{\hat{\Omega}} G(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1}\left(\int_{-1}^{1} G(\xi,\eta)\,d\xi\right)d\eta$$

$$= \sum_{m=1}^{N_i}\left(\sum_{n=1}^{N_i} G(\xi_n,\eta_m)w_n\right)w_m = \sum_{l=1}^{N_l} G(\xi_l,\eta_l)w_l$$

- Here we re-labeled *(m,n)* with a single index $l=1$



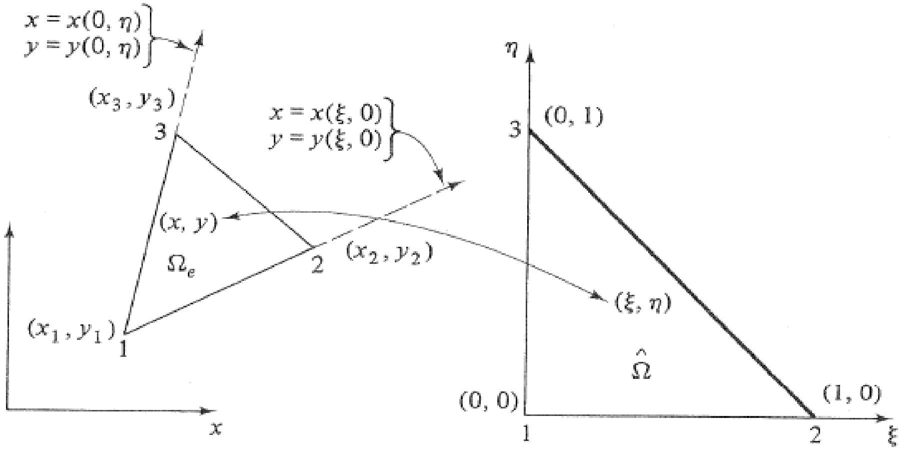
Triangular elements

- We first consider triangular with straight sides. We consider the mapping from a right-isosceles master triangle. By inspection, we can write the basis functions as:

$$\widehat{N}_1(\xi,\eta) = 1 - \xi - \eta$$

$$\widehat{N}_2(\xi,\eta) = \xi$$

$$\widehat{N}_3(\xi,\eta) = \eta$$



- The coordinate mapping is then defined from:

$$x = \sum_{j=1}^{3} x_j \widehat{N}_j(\xi,\eta)$$

$$y = \sum_{j=1}^{3} y_j \widehat{N}_j(\xi,\eta),$$

Triangular elements

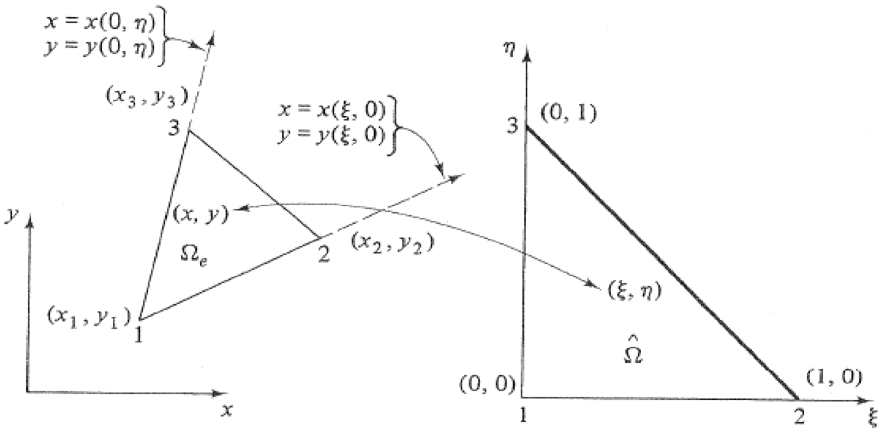
$$x = \sum_{j=1}^{3} x_j \widehat{N}_j(\xi,\eta)$$

$$y = \sum_{j=1}^{3} y_j \widehat{N}_j(\xi,\eta),$$

$$\widehat{N}_1(\xi,\eta) = 1-\xi-\eta$$

$$\widehat{N}_2(\xi,\eta) = \xi$$

$$\widehat{N}_3(\xi,\eta) = \eta$$



- Inverting this mapping gives:

$$\xi = \frac{1}{2A_e}\{(y_3-y_1)(x-x_1)-(x_3-x_1)(y-y_1)\}$$

$$\eta = \frac{1}{2\underbrace{A_e}_{\substack{area\\ of\ \Omega_e}}}\{-(y_2-y_1)(x-x_1)+(x_2-x_1)(y-y_1)\}$$

- Can you recognize these as the linear shape functions of the 4 node quadrilateral element? (take nodes $3 \equiv 4$)

$$\xi = N_2^e(x,y)$$

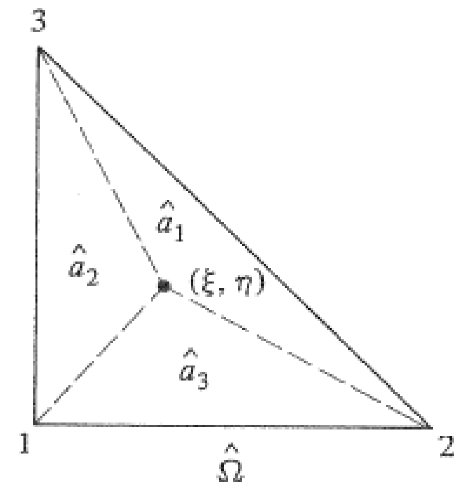
$$\eta = N_3^e(x,y)$$

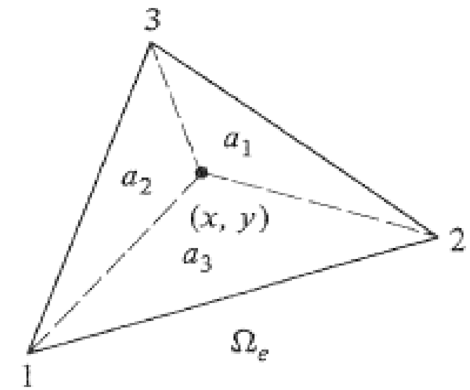
$$1-\xi-\eta = N_1^e(x,y)$$

Using these, one can now easily compute $\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}, |J|$ and thus the element stiffness and load

Area coordinates

- The expressions for $\xi, \eta, 1-\xi-\eta$ can easily be interpreted as ratios of areas. We will see this interpretation to be useful in deriving higher order triangular elements.
- Let us join the points (ξ,η) and *(x,y)* with the vertices of the triangles $\hat{\Omega}$ and $\Omega_e,$ respectively. We denote $\hat{a}_i, a_i$ as the areas of the subtriangles opposite node in $\Omega_e,$ and $\hat{\Omega},$ respectively.
- We define the area coordinates on $\hat{\Omega}$ as: $\zeta_i = \hat{a}_i / \hat{A}, i=1,2,3,$ where $\hat{A}=1/2$ is the area of the master element.





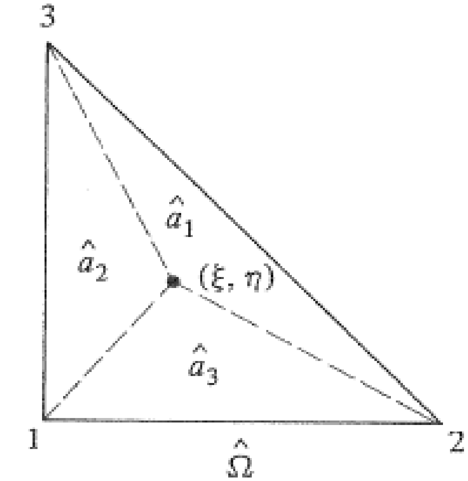
$$\zeta_1 = 1-\xi-\eta$$
$$\zeta_2 = \xi$$
$$\zeta_3 = \eta$$

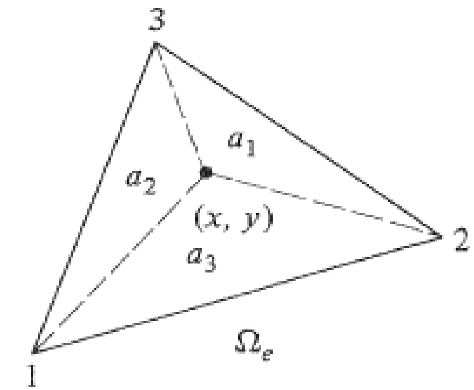
Area coordinates

- Since |J| is constant (the ratio of the areas of Ω_e, and $\hat{\Omega}$,), the map T_e transforms areas uniformly, thus:

$$\zeta_i = \frac{\hat{a}_i}{\hat{A}} = \frac{a_i}{A_e}, i = 1, 2, 3,$$

- This is only true for triangles with straight sides.

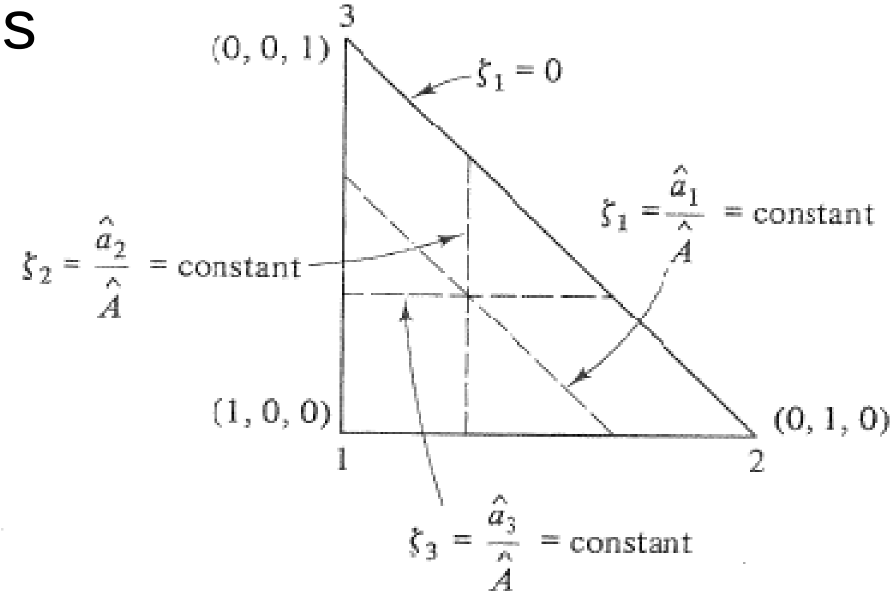


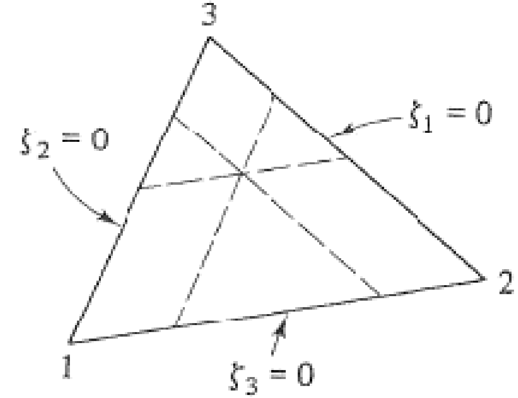


$$\zeta_1 = 1 - \xi - \eta$$
$$\zeta_2 = \xi$$
$$\zeta_3 = \eta$$

Area coordinates

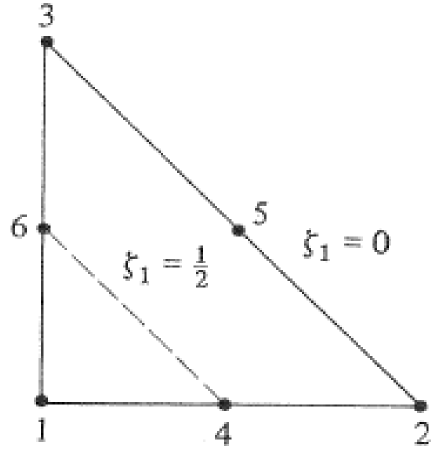
- Several interesting properties of area coordinates are shown in the figures.
- At a given point, the line $\zeta_i = constant$ is parallel to the side of the element opposite node i.
- The boundary segments of the element are defined by $\zeta_i = 0, i = 1,2,3.$
- The vertices of the triangle are (1,0,0), (0,1,0) and (0,0,1).





Higher-degree shape functions

- The area coordinates ζ_i on $\widehat{\Omega}$, can be used to determine higher degree shape functions $\widehat{\psi}_i(\zeta)$.

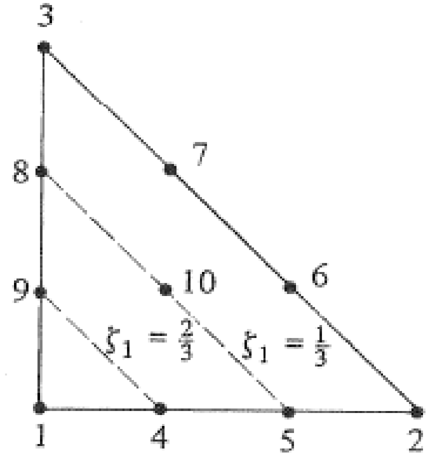


$$\widehat{N}_1 = 2\zeta_1(\zeta_1 - \frac{1}{2}),\ \widehat{N}_4 = 4\zeta_1\zeta_2$$

$$\widehat{N}_2 = 2\zeta_2(\zeta_2 - \frac{1}{2}),\ \widehat{N}_5 = 4\zeta_2\zeta_3$$

$$\widehat{N}_3 = 2\zeta_3(\zeta_3 - \frac{1}{2}),\ \widehat{N}_6 = 4\zeta_3\zeta_1$$

Quadratic shape functions



$$\widehat{N}_1 = \frac{9}{2}\zeta_1(\zeta_1 - \frac{2}{3})(\zeta_1 - \frac{1}{3})$$

$$\widehat{N}_4 = \frac{27}{2}\zeta_1\zeta_2(\zeta_1 - \frac{1}{3})$$

$$\widehat{N}_3 = \frac{27}{2}\zeta_1\zeta_2(\zeta_2 - \frac{1}{3})$$

$$\widehat{N}_{10} = 27\zeta_1\zeta_2\zeta_3$$

Cubic shape functions

Shape functions on triangles using area coordinates

- The coordinate transformation is now having the form:

$$x(\xi,\eta) \equiv x(\zeta_1,\zeta_2,\zeta_3) = \sum_{j=1}^{N_e} x_j \widehat{N}_j(\zeta_1,\zeta_2,\zeta_3)$$

$$y(\xi,\eta) \equiv y(\zeta_1,\zeta_2,\zeta_3) = \sum_{j=1}^{N_e} y_j \widehat{N}_j(\zeta_1,\zeta_2,\zeta_3),$$

- The calculations here defer from those used in quadrilaterals because of the redundant area coordinate $\zeta_1 = 1 - \zeta_2 - \zeta_3$.
- Calculation of derivatives proceeds as:

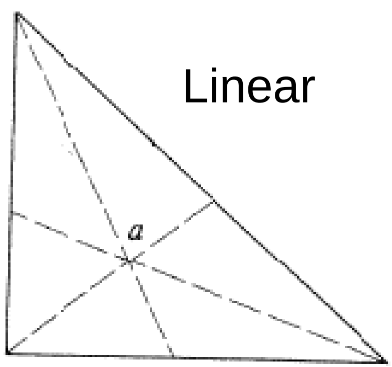
$$\frac{\partial \widehat{N}_1}{\partial x} = \frac{\partial \widehat{N}_1}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial x} + \frac{\partial \widehat{N}_1}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial x} + \frac{\partial \widehat{N}_1}{\partial \zeta_3}\frac{\partial \zeta_3}{\partial x}$$

- Alternatively, one can use $\zeta_1 = 1 - \zeta_2 - \zeta_3, \zeta_2 \equiv \xi, \zeta_3 \equiv \eta,$ and proceed exactly as was done before for quadrilaterals.

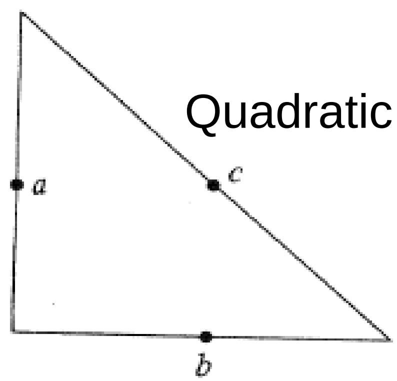
Quadrature integration formulas for triangles

- We use particular quadrature rules appropriate for the area coordinates introduced earlier.

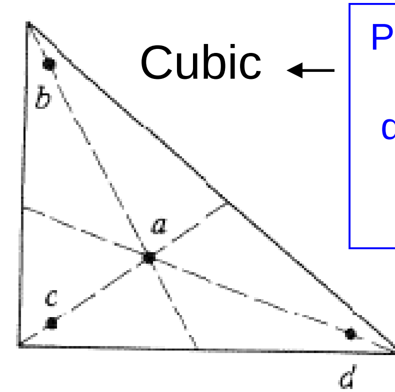
$$\int_{\hat{\Omega}} G(\zeta_1,\zeta_2,\zeta_3)\,d\zeta_2\,d\zeta_3 \ (note\ \zeta_1 = 1-\zeta_2-\zeta_3) = \sum_{l=1}^{N_{int}} G(\underbrace{\zeta_{1l},\zeta_{2l},\zeta_{3l}}_{\text{quadrature points in } \hat{\Omega}}) \underbrace{w_l}_{\text{integration weights}}$$



points	ξ_l	w_l
a	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\frac{1}{2}$



points	ξ_l	w_l
a	$(\frac{1}{2}, 0, \frac{1}{2})$	$\frac{1}{6}$
b	$(\frac{1}{2}, \frac{1}{2}, 0)$	$\frac{1}{6}$
c	$(0, \frac{1}{2}, \frac{1}{2})$	$\frac{1}{6}$



← Polynomials up to this degree are integrated exactly

points	ξ_l	w_l
a	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$-\frac{27}{96}$
b	$(\frac{2}{15}, \frac{2}{15}, \frac{11}{15})$	$\frac{25}{96}$
c	$(\frac{11}{15}, \frac{2}{15}, \frac{2}{15})$	$\frac{25}{96}$
d	$(\frac{2}{15}, \frac{11}{15}, \frac{2}{15})$	$\frac{25}{96}$