MAE4700/5700

Finite Element Analysis for Mechanical and Aerospace Design

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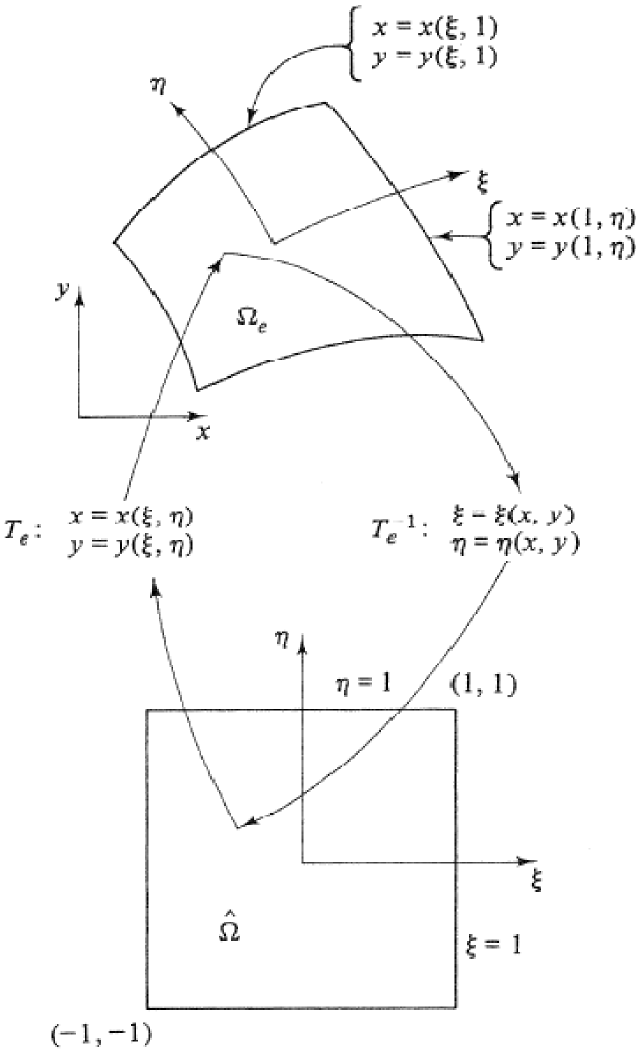
Two-dimensional element calculations

- Similarly to 1D, we here also define the concept of a master element in natural coordinates (ξ, η)
- Performing calculations on elements with curved sides as shown is difficult in the (x, y) coordinates.
  - For example, the limits of integration will vary from element to element

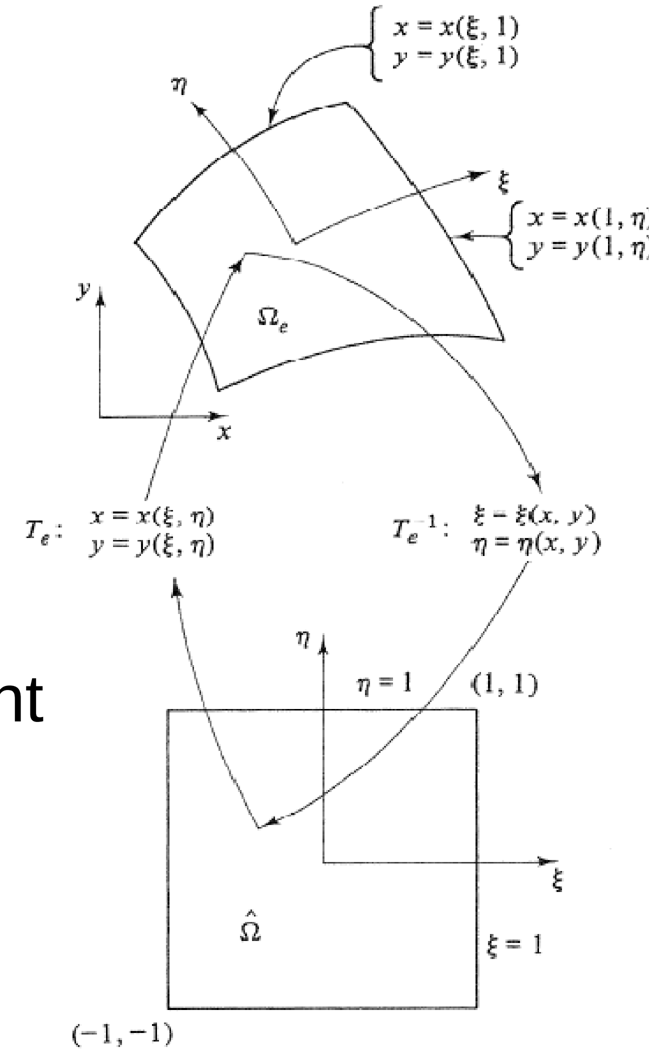
Can we find a master element $\hat{\Omega}$ that is mapped appropriately to each element Ω_e ? Then with transformation $\hat{\Omega} \rightarrow \Omega_e$ we can perform all integrations for all elements at the master element.



Two-dimensional element calculations

- We consider the master element to be a square with: $-1 \le \xi, \eta \le 1$
- We define the map $\hat{\Omega} \rightarrow \Omega_e$ as follows

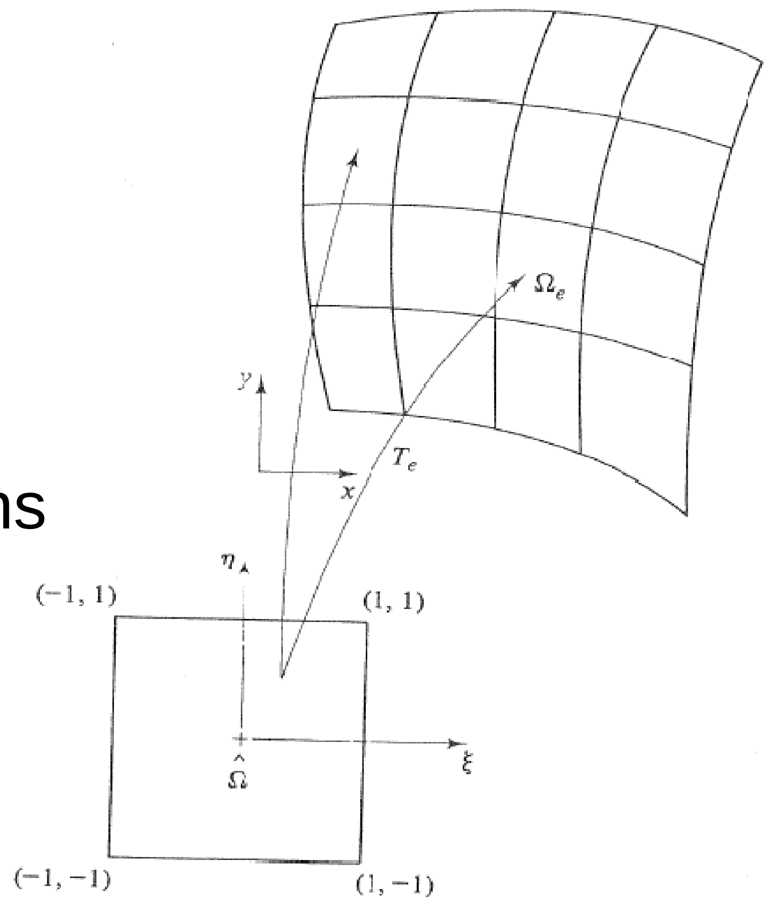
$$T_e : \begin{array}{l} x = x(\xi, \eta) \\ y = y(\xi, \eta) \end{array}$$



- Note that the boundary segment $\xi = 1$ of $\hat{\Omega}$ is mapped on the curved segment $x = x(1, \eta)$, $y = y(1, \eta)$ with η as a parameter. Similarly for the other 3 boundary segments.

Two-dimensional element calculations

- The complete finite element mesh is seen as a sequence of transformations $\{T_1, T_2, ..., T_E\}$ with each of them mapping $T_e : \widehat{\Omega} \to \Omega_e, e = 1, 2, ..., E$



- We need to figure out how we can transfer all needed calculations in each element Ω_e to the master element $\widehat{\Omega}$.
- The plan is to make all these calculations defined in terms of the mappings $\{T_1, T_2, ..., T_E\}$.

Transformation equations

- We assume that the mappings

$$x = x(\xi,\eta), y = y(\xi,\eta)$$

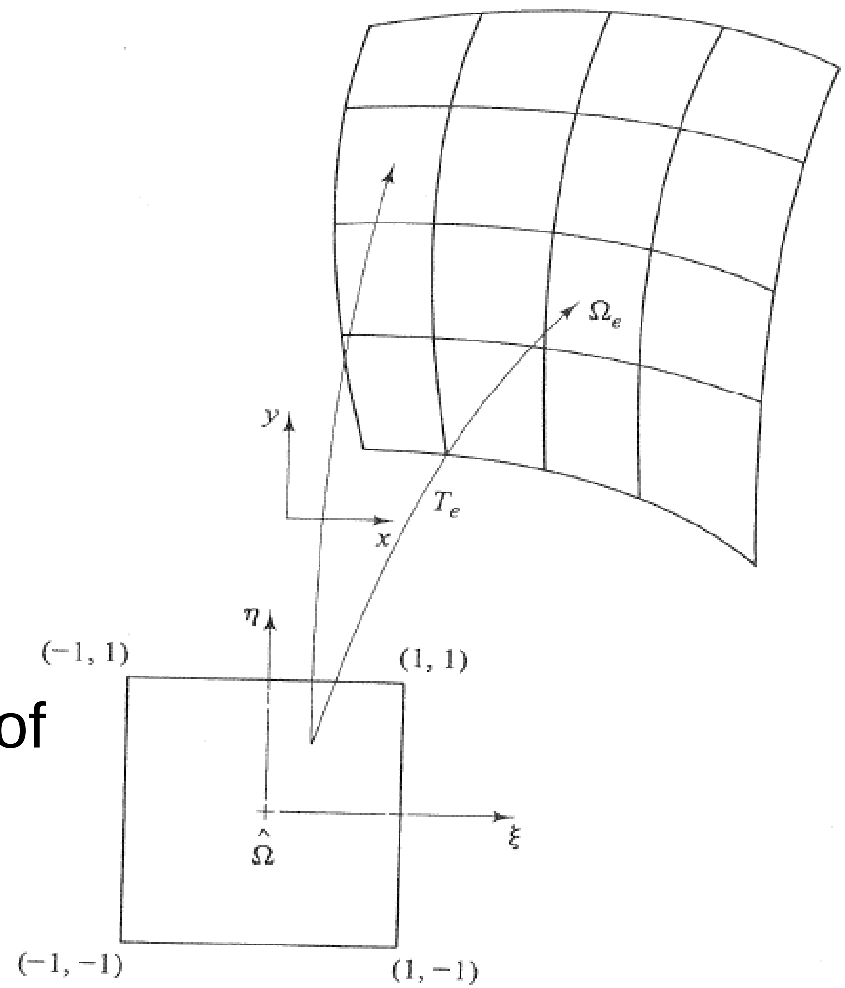
are differentiable. We can then write:

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta,$$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$$

- We can write this as a system of Eqs:

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}_{Jacobian\ matrix\ [J]^T} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix}$$



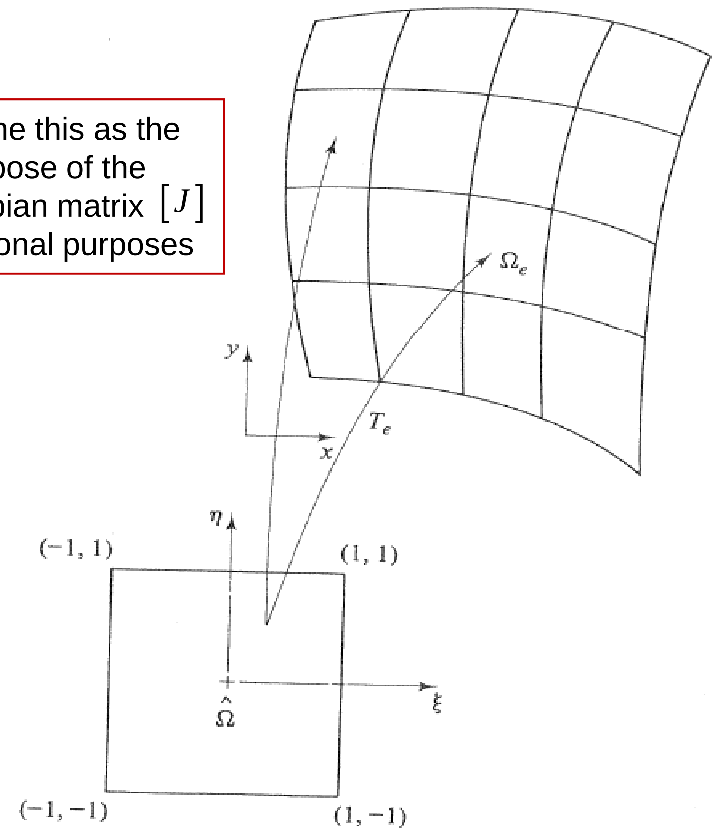
Transformation equations

- The Jacobian maps segments

  $d\xi, d\eta$ from $\hat{\Omega}$ to Ω_e .

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}_{Jacobian\ matrix\,[J]^T} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix}$$

We define this as the transpose of the Jacobian matrix $[J]$ for notational purposes



- Assume the inverse transformation exists:

$$T_e^{-1} : \Omega_e \rightarrow \hat{\Omega}, e = 1, 2, \ldots, E$$

$$\begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial x}{\partial \eta} \\ -\dfrac{\partial y}{\partial \xi} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} dx \\ dy \end{Bmatrix}$$

Determinant of the Jacobian matrix

$$|J| = \det[J] = \det[J]^T = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

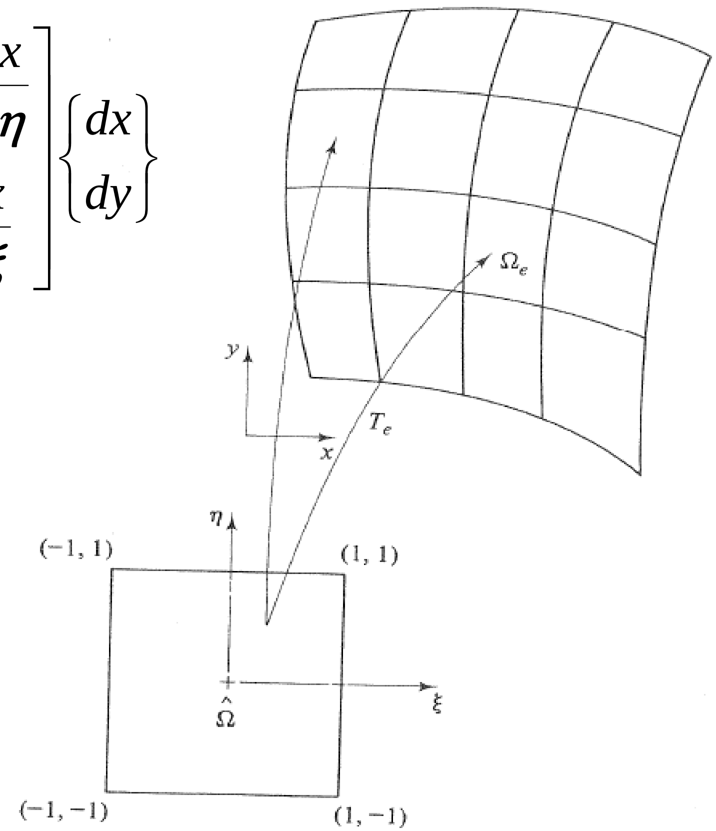
Transformation equations

$$\xi = \xi(x, y)$$

$$T_e^{-1}:$$

$$\eta = \eta(x, y)$$

$$\begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix}\begin{Bmatrix} dx \\ dy \end{Bmatrix}$$



- From the equations above we can conclude that:

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta}, \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial \eta},$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi}, \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial \xi}$$

Constructing the transformation T_e

- We are interested for differentiable mappings $\{T_1, T_2, \ldots, T_E\}$ of the form $x = x(\xi,\eta), y = y(\xi,\eta)$ for each element that do not create gaps or overlapping between elements and which are easy to construct from the geometry of each element Ω_e .
- Based on an earlier lecture on construction of element basis functions, suppose that we have in place element basis functions $\widehat{N}_j(\xi,\eta)$ in $\widehat{\Omega}$ such that any function $\widehat{g}(\xi,\eta)$ can be approximated as:

$$\widehat{g}(\xi,\eta) = \sum_{j=1}^{M} g_j \widehat{N}_j(\xi,\eta), \;\; g_j = \widehat{g}(\underbrace{\xi_j,\eta_j}_{\substack{coordinates\ of\ node\ j \\ \text{in the master element}}})$$

where *M* is the number of nodes in the master element.

Constructing the transformation T_e

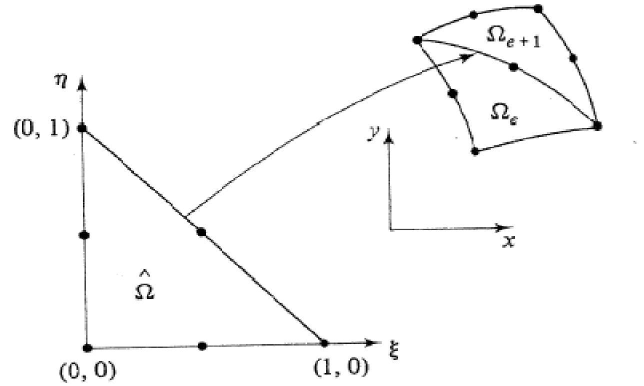
- The finite element basis functions can be used to construct the mapping T_e as follows:

$$x = \sum_{j=1}^{M} x_j \widehat{N}_j(\xi,\eta),$$

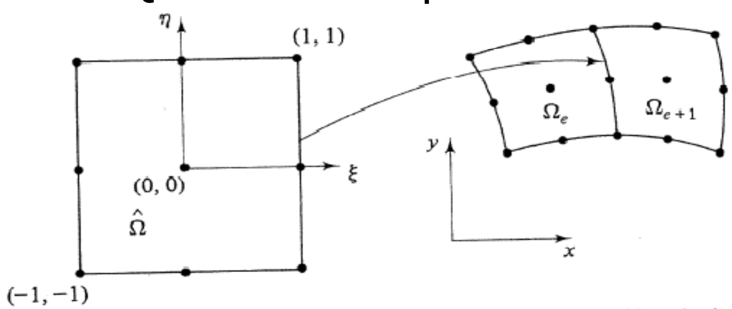
$$y = \sum_{j=1}^{M} y_j \widehat{N}_j(\xi,\eta),$$

where (x_j, y_j) are the coordinates of node *j* in the element Ω_e .

- Can you convince yourselves that this mapping does not create gaps or element overlapping?



Quadratic shape functions



Biquadratic shape functions

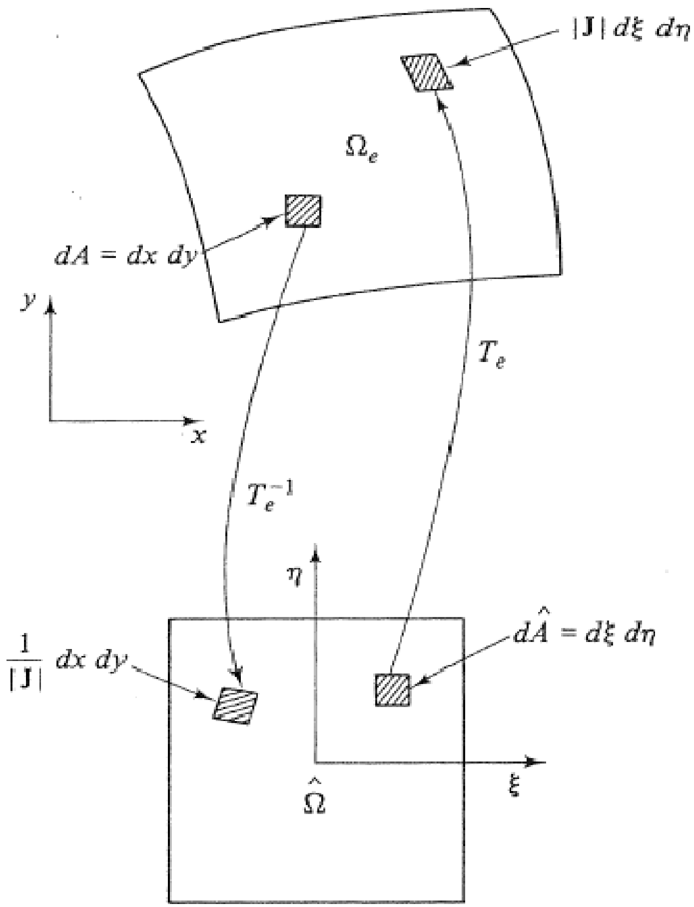
Note that the common side between elements e and e+1 is uniquely defined in terms of the nodal coordinates that are shared by the 2 elements.

Geometric interpretation of the Jacobian

- The value of |J| is the ratio of areas of elements at points $x = x(\xi,\eta), y = y(\xi,\eta)$ and (ξ,η).

$$dA = |J|\, d\hat{A}$$

- We need to select the element basis functions $\hat{N}_i$ and (x_i, y_i) such that |J|>0 (area cannot be transformed to a line or a point or turned inside out!).
- This guarantees that the mapping T_e is invertable.

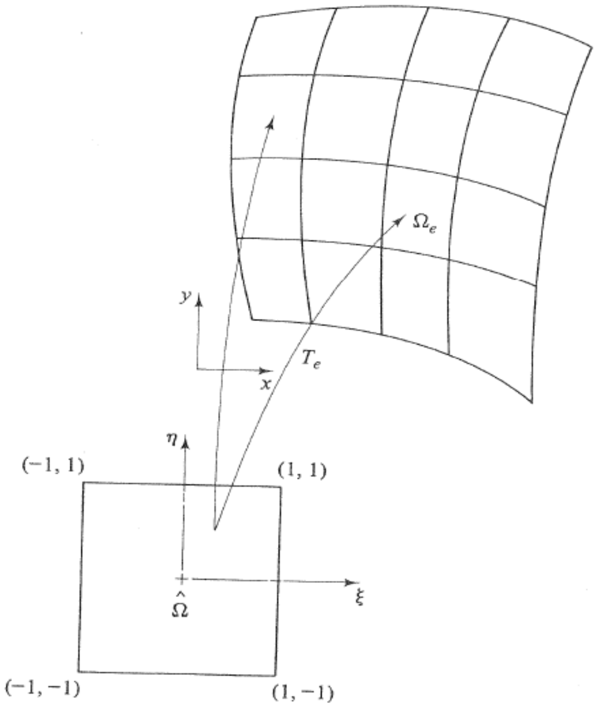


Transformation equations

- Let us return to the definition of $|J|$ and the earlier stated transformation equations:

$$|J| = \det J = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial \eta},\ \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial \eta},\ \frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial \xi},\ \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial \xi}$$



- Applying the mapping:

$$x = \sum_{j=1}^{M} x_j \widehat{N}_j(\xi,\eta),\ y = \sum_{j=1}^{M} y_j \widehat{N}_j(\xi,\eta),$$

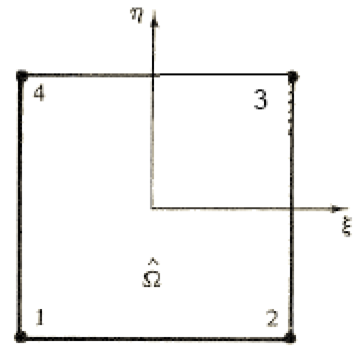
leads to:

$$|J| = \det J = \left(\sum_{j=1}^{M} x_j \frac{\partial \widehat{N}_j(\xi,\eta)}{\partial \xi}\right)\left(\sum_{j=1}^{M} y_j \frac{\partial \widehat{N}_j(\xi,\eta)}{\partial \eta}\right) - \left(\sum_{j=1}^{M} x_j \frac{\partial \widehat{N}_j(\xi,\eta)}{\partial \eta}\right)\left(\sum_{j=1}^{M} y_j \frac{\partial \widehat{N}_j(\xi,\eta)}{\partial \xi}\right)$$

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\sum_{j=1}^{M} y_j \frac{\partial \widehat{N}_j(\xi,\eta)}{\partial \eta},\ \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\sum_{j=1}^{M} x_j \frac{\partial \widehat{N}_j(\xi,\eta)}{\partial \eta},\ \frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\sum_{j=1}^{M} y_j \frac{\partial \widehat{N}_j(\xi,\eta)}{\partial \xi},\ \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\sum_{j=1}^{M} x_j \frac{\partial \widehat{N}_j(\xi,\eta)}{\partial \eta}$$

Example problem

- Consider the following master element.



- The basis functions for this element are:

$$\widehat{N}_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$$

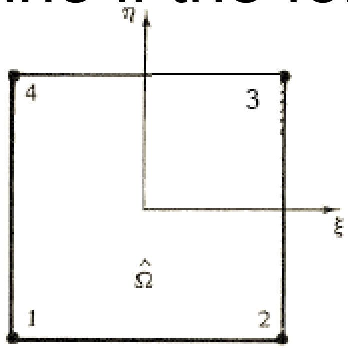
$$\widehat{N}_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$$

$$\widehat{N}_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$$

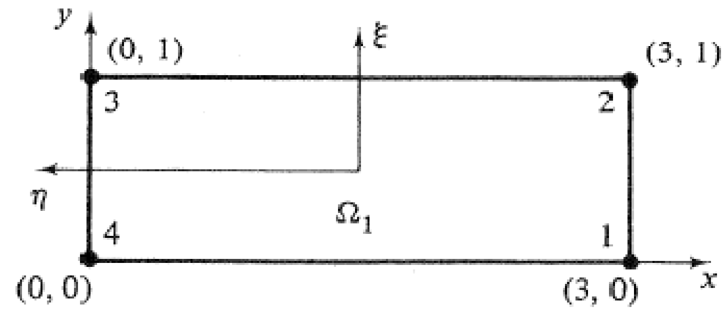
$$\widehat{N}_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$

Example problem

- Examine if the following mapping is invertable.



$$T_1 \longrightarrow \qquad T_1^{-1}? \longleftarrow$$



$$x = \sum_{j=1}^{M} x_j \widehat{N}_j(\xi,\eta) = 3\widehat{N}_1(\xi,\eta) + 3\widehat{N}_2(\xi,\eta) = \frac{3}{2}(1-\eta)$$

$T_1:$

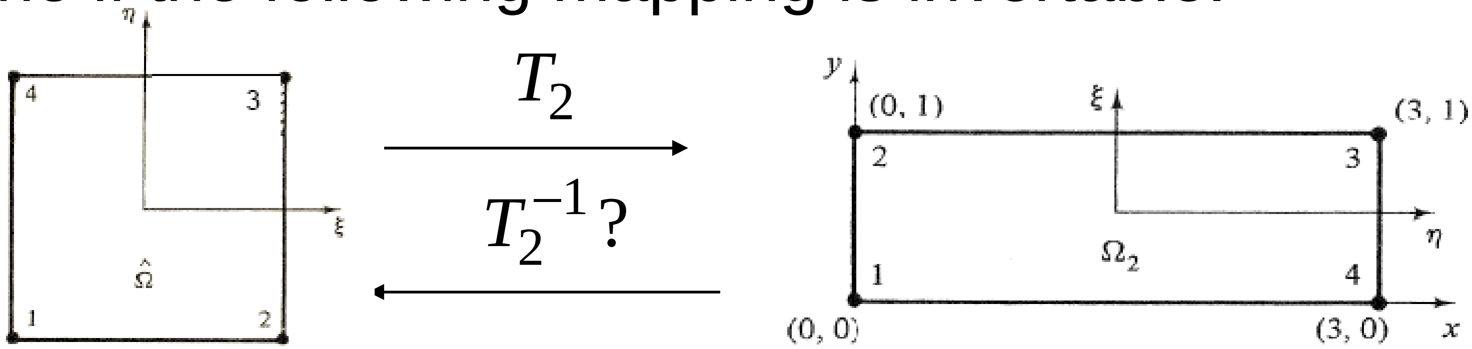
$$y = \sum_{j=1}^{M} y_j \widehat{N}_j(\xi,\eta) = \widehat{N}_2(\xi,\eta) + \widehat{N}_3(\xi,\eta) = \frac{1}{2}(1+\xi)$$

- Note that $|J| = \det\begin{bmatrix} 0 & -\frac{3}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = \frac{3}{4} > 0 \Rightarrow$ the map T_1 is invertable.

- |J| is the ratio of the area of Ω_1 to area of $\hat{\Omega}$.

Example problem

- Examine if the following mapping is invertable.



$$x = \sum_{j=1}^{M} x_j \widehat{N}_j(\xi,\eta) = 3\widehat{N}_3(\xi,\eta) + 3\widehat{N}_4(\xi,\eta) = \frac{3}{2}(1+\eta)$$

T_2 :

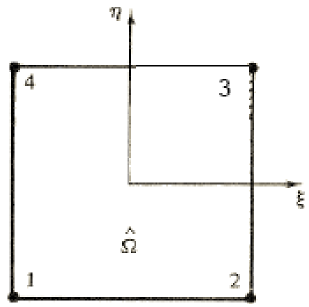
$$y = \sum_{j=1}^{M} y_j \widehat{N}_j(\xi,\eta) = \widehat{N}_2(\xi,\eta) + \widehat{N}_3(\xi,\eta) = \frac{1}{2}(1+\xi)$$

- Note that $|J| = \det\begin{bmatrix} 0 & \frac{3}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = -\frac{3}{4} < 0 \Rightarrow$ the map T_2 is not invertable.

- The clockwise numbering of nodes should be avoided.

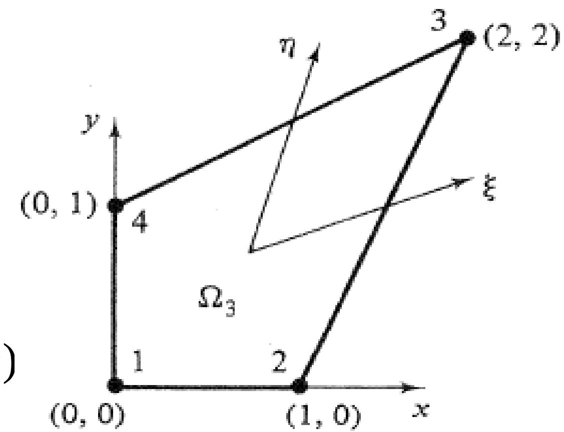
Example problem

- Examine if the following mapping is invertable.



$$\xrightarrow{T_3}$$

$$\xleftarrow{T_3^{-1}\,?}$$



T_3 :

$$x = \sum_{j=1}^{M} x_j \widehat{N}_j(\xi,\eta) = \widehat{N}_2(\xi,\eta) + 2\widehat{N}_3(\xi,\eta) = \frac{1}{4}(3 + 3\xi + \eta + \xi\eta)$$

$$y = \sum_{j=1}^{M} y_j \widehat{N}_j(\xi,\eta) = 2\widehat{N}_3(\xi,\eta) + \widehat{N}_4(\xi,\eta) = \frac{1}{4}(3 + \xi + 3\eta + \xi\eta)$$

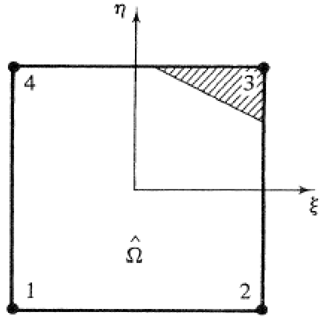
$$|J| = \det\begin{bmatrix} \frac{1}{4}(3+\eta) & \frac{1}{4}(1+\xi) \\ \frac{1}{4}(1+\eta) & \frac{1}{4}(3+\xi) \end{bmatrix} = \frac{1}{2} + \frac{1}{8}\xi + \frac{1}{8}\eta \Rightarrow$$

|J| cannot be zero inside the element so the map T_3 is invertable.

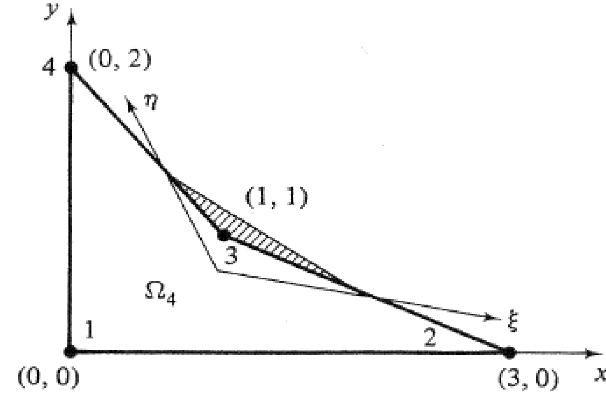
- |J| is smaller near node 1, largest near node 3.

Example problem

- Examine if the following mapping is invertable.



$$T_4 \longrightarrow \qquad \longleftarrow T_4^{-1}?$$



$T_4 :$

$$x = \sum_{j=1}^{M} x_j \widehat{N}_j(\xi,\eta) = 3\widehat{N}_2(\xi,\eta) + \widehat{N}_3(\xi,\eta) = \frac{1}{4}(4+4\xi-2\eta-2\xi\eta)$$

$$y = \sum_{j=1}^{M} y_j \widehat{N}_j(\xi,\eta) = \widehat{N}_3(\xi,\eta) + 2\widehat{N}_4(\xi,\eta) = \frac{1}{4}(3-\xi+3\eta-\xi\eta)$$

$$|J| = \det\begin{bmatrix} \frac{1}{4}(4-2\eta) & -\frac{1}{4}(2+2\xi) \\ -\frac{1}{4}(1+\eta) & \frac{1}{4}(3-\xi) \end{bmatrix} = \frac{1}{8}(5-3\xi-4\eta) \Rightarrow$$

|J| is not >0 everywhere inside the element so the map T_3 is not invertable. The marked region near node 3 -- above the line $\xi = 5/3 - 4/3\,\eta$ -- is mapped outside the element.

- All angles of quadrilateral elements need to be < π.

Finite element equations

- To solve our boundary value problem of interest examined earlier, we need to compute the following matrices and vectors ($1 \le e \le E, 1 \le i, j \le N_e$):

$$k_{ij}^e = \int_{\Omega_e} \left[k \left(\frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} \right) + b N_i^e N_j^e \right] dxdy \qquad f_i^e = \int_{\Omega_e} f N_i^e dxdy$$

$$P_{ij}^e = \int_{\partial\Omega_{2h}^e} p N_i^e N_j^e ds \qquad \gamma_i^e = \int_{\partial\Omega_{2h}^e} \gamma N_i^e ds$$

- *k* and *f* are assumed to be functions of *(x,y),* whereas *p* and γ are assumed functions of *s* defining $\partial\Omega_{2h}(x(s), y(s))$
- It remains to discuss how to compute these using the master element and the transformation T_e.

Master element calculations: Step 1

- We construct the mappings $T_e : \widehat{\Omega} \rightarrow \Omega_e, e = 1, 2, ..., E$ as follows:

$$x = \sum_{j=1}^{N_e} x_j \widehat{N}_j(\xi, \eta)$$

$$y = \sum_{j=1}^{N_e} y_j \widehat{N}_j(\xi, \eta),$$

- This mapping $T_e : \widehat{\Omega} \rightarrow \Omega_e,$ is completed defined by the nodal coordinates (x_j, y_j) of element *e*.

Master element calculations: Step 2

- Let *g(x,y)* be any function of *(x,y)* in Ω_e. We can convert *g* to a function of (ξ,η) defined in $\hat{\Omega}$ as follows:

$$g(x,y) = g(x(\xi,\eta), y(\xi,\eta)) \equiv \hat{g}(\xi,\eta)$$

- The transformation $x(\xi,\eta), y(\xi,\eta)$ is defined in Step 1.
- The element shape functions $N_j^e(x,y)$ are simply obtained from $\hat{N}_j(\xi,\eta)$ as follows:

$$N_j^e(x,y) = \hat{N}_j(x(\xi,\eta), y(\xi,\eta)), \quad j = 1,2,\ldots,N_e$$

Master element calculations: Step 2

- The derivatives of $N_j^e(x,y)$ with respect to *x* and *y* are:

$$\frac{\partial N_j^e(x,y)}{\partial x} = \frac{\partial \widehat{N}_j}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \widehat{N}_j}{\partial \eta}\frac{\partial \eta}{\partial x},$$

$$\frac{\partial N_j^e(x,y)}{\partial y} = \frac{\partial \widehat{N}_j}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \widehat{N}_j}{\partial \eta}\frac{\partial \eta}{\partial y}$$

where:

$$\frac{\partial x}{\partial \xi} = \sum_{k=1}^{N_e} x_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \xi}, \quad \frac{\partial x}{\partial \eta} = \sum_{k=1}^{N_e} x_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \eta}$$

$$\frac{\partial y}{\partial \eta} = \sum_{k=1}^{N_e} y_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \eta}, \quad \frac{\partial y}{\partial \eta} = \sum_{k=1}^{N_e} y_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \eta}$$

- Finally, with |J| the Jacobian of T_e, using

$$\frac{\partial \xi}{\partial x} = \frac{1}{|J|}\sum_{k=1}^{M} y_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \eta}, \quad \frac{\partial \xi}{\partial y} = -\frac{1}{|J|}\sum_{k=1}^{M} x_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \eta}, \quad \frac{\partial \eta}{\partial x} = -\frac{1}{|J|}\sum_{k=1}^{M} y_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \xi}, \quad \frac{\partial \eta}{\partial y} = \frac{1}{|J|}\sum_{k=1}^{M} x_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \eta}$$

we conclude:

$$\frac{\partial N_j^e(x,y)}{\partial x} = \frac{1}{|J|}\left\{\frac{\partial \widehat{N}_j}{\partial \xi}\sum_{j=1}^{M} y_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \eta} - \frac{\partial \widehat{N}_j}{\partial \eta}\sum_{k=1}^{M} y_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \xi}\right\},$$

$$\frac{\partial N_j^e(x,y)}{\partial y} = \frac{1}{|J|}\left\{-\frac{\partial \widehat{N}_j}{\partial \xi}\sum_{k=1}^{M} x_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \eta} + \frac{\partial \widehat{N}_j}{\partial \eta}\sum_{jk=1}^{M} x_k \frac{\partial \widehat{N}_k(\xi,\eta)}{\partial \eta}\right\}$$

Note regarding convergence rates

- We had presented earlier that the asymptotic rates of convergece are only valid if $w_h^e(x,y)=\sum_{j=1}^{N_e} w_j N_j^e(x,y)$

  contains complete polynomials of degree *k* for smooth interpolated functions.
- However, with the transformation

$$N_j^e(x,y)=\widehat{N}_j(x(\xi,\eta),y(\xi,\eta)),\quad j=1,2,\ldots,N_e$$

  the shape functions $N_j^e(x,y)$ may not even be polynomials!
- It can be shown that if the master basis functions $\widehat{N}_j(\xi,\eta)$ contain complete polynomials of degree *k* and *|J|>0*, the error estimates still hold.

Master element calculations: Step 3

- The domain integrals of interest now take the form:

$$\int_{\Omega_e} g(x,y)dxdy = \int_{\widehat{\Omega}} g(x(\xi,\eta),y(\xi,\eta))\,|J|\,d\xi d\eta \equiv \int_{\widehat{\Omega}} \underbrace{\widehat{g}(\xi,\eta)\,|J|}_{\widehat{G}(\xi,\eta)}d\xi d\eta$$

- Our integrals are now in the domain of $\widehat{\Omega}$

$$\int_{\Omega_e} g(x,y)dxdy = \int_{\widehat{\Omega}} g(x(\xi,\eta),y(\xi,\eta))\,|J|\,d\xi d\eta \equiv \int_{\widehat{\Omega}} \widehat{G}(\xi,\eta)d\xi d\eta$$

- As in 1D, we use Gauss integration (to be discussed):

$$\int_{\widehat{\Omega}} \widehat{G}(\xi,\eta)d\xi d\eta \approx \sum_{i=1}^{N_i} \widehat{G}(\underbrace{\xi_i,\eta_i}_{\substack{Gauss\\ \text{integration}\\ \text{points}}})\underbrace{w_i}_{\substack{Gauss\\ \text{integration}\\ \text{weights}}}$$

Master element calculations: Step 4

- In numerical calculation of e.g. the element stiffness,

$$k_{ij}^e = \int_{\Omega_e} \left[k \left(\frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} \right) + b N_i^e N_j^e \right] dxdy$$

  we often need to consider functions *k(x,y). It is common to use the finite element interpolant to approximate this function*, i.e.

$$k_h(x,y) = \sum_{j=1}^{N_e} \underbrace{k_j}_{k(x_j,y_j)} N_j^e(x,y) = \sum_{j=1}^{N_e} k_j \widehat{N}_j(\xi,\eta)$$

  This requires only the nodal values of *k*. Similar calculation can be done for integrals involving *b(x,y)* and *f(x,y)*.

Master element calculations: Step 5

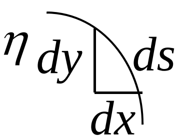
- We need to consider a number of integrals that involve boundary integration:

$$P_{ij}^e = \int_{\partial\Omega_{2h}^e} p N_i^e N_j^e ds \qquad \gamma_i^e = \int_{\partial\Omega_{2h}^e} \gamma N_i^e ds$$

- Lets introduce the restriction of the basis functions $\widehat{N}_j(\xi,\eta)$ on the side $\xi = 1$. We denote them as: $\hat{\theta}(\eta) \equiv \widehat{N}_j(1,\eta),\ j = 1,2,..,N_e$. These functions are only non-zero for nodes on the boundary $\xi = 1$. The 2 integrals above can be computed as:

$|j(\eta)|$ is the Jacobian of the transformation of η to s

$$ds = |j(\eta)|\, d\eta$$



$$ds = \left[dx^2 + dy^2\right]^{1/2}$$

$$P_{ij}^e = \int_{\partial\Omega_{2h}^e} p N_i^e N_j^e ds = \int_{-1}^{1} \hat{p}(\eta)\hat{\theta}_i(\eta)\hat{\theta}_j(\eta)\,|j(\eta)|\,d\eta$$

$$\gamma_i^e = \int_{\partial\Omega_{2h}^e} \gamma N_i^e ds = \int_{-1}^{1} \hat{\gamma}(\eta)\hat{\theta}_i(\eta)\,|j(\eta)|\,d\eta$$

$$|j(\eta)| = \left[\left(\frac{\partial x(1,\eta)}{\partial\eta}\right)^2 + \left(\frac{\partial y(1,\eta)}{\partial\eta}\right)^2\right]^{1/2}$$

Notes: (a) Use FE interpolants of $\hat{p}(\eta), \hat{\gamma}(\eta)$ (b) Use 1D Gauss integration.

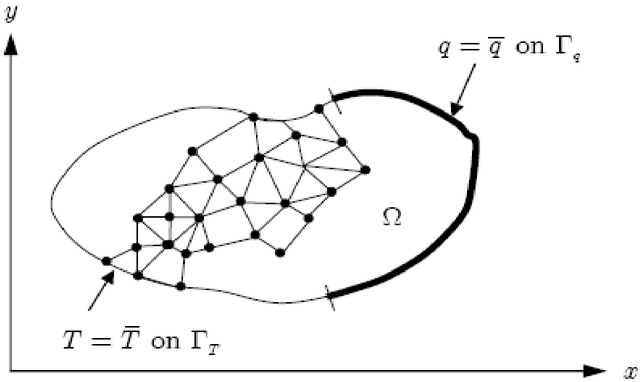
Example problem using 3-node triangular elements

- Consider 2D heat conduction using the general problem discussed in an earlier lecture.
- We are computing *T(x,y) such that*:

$$-\nabla \bullet (D\nabla T) = f(x,y)$$

$$T = \overline{T} \text{ on } \Gamma_T$$

$$q = -D\nabla T \bullet n = \overline{q} \text{ on } \Gamma_q$$



- Recall that the final weak form is:

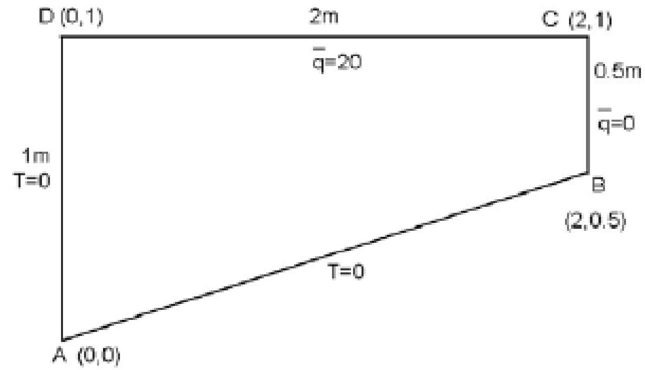
$$\{w\}^T \sum_{e=1}^{nel} [L^e]^T \underbrace{\int_{\Omega^e} [B^e]^T [D^e][B^e] d\Omega}_{K^e} [L^e]\{d\} = \{w\}^T \sum_{e=1}^{nel} [L^e]^T \left(\underbrace{\int_{\Omega^e} [N^e]^T f \, d\Omega}_{\{f_\Omega^e\}} - \underbrace{\int_{\Gamma_q^e} [N^e]^T \overline{q} d\Gamma}_{\{f_\Gamma^e\}} \right), \forall \{w_F\}$$

$$[B^e] = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \cdots & \frac{\partial N_{nen}^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \cdots & \frac{\partial N_{nen}^e}{\partial y} \end{bmatrix} \quad [N^e] = \begin{bmatrix} N_1^e & N_2^e & \cdots & N_{nen}^e \end{bmatrix}$$

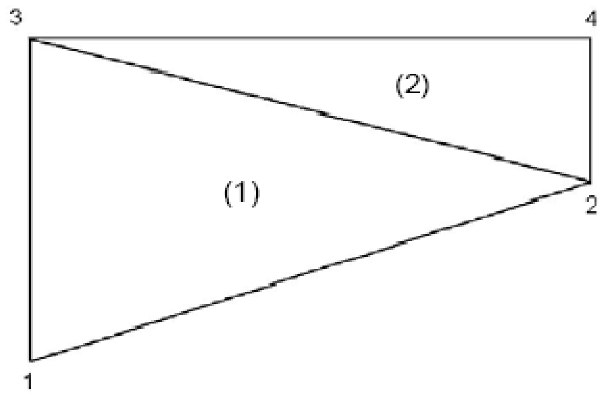
$$\begin{bmatrix} K_E & K_{EF} \\ K_{FE} & K_F \end{bmatrix} \begin{bmatrix} \overline{d}_E \\ d_F \end{bmatrix} = \begin{bmatrix} f_E + r_E \\ f_F \end{bmatrix}$$

Example problem using 3-node triangular elements

- Consider the heat conduction problem in the figure. We assume isotropic conductivity $k=5\ W^{o}C^{-1}$ and a source term f=6 Wm^{-2}. The BCs are as shown.

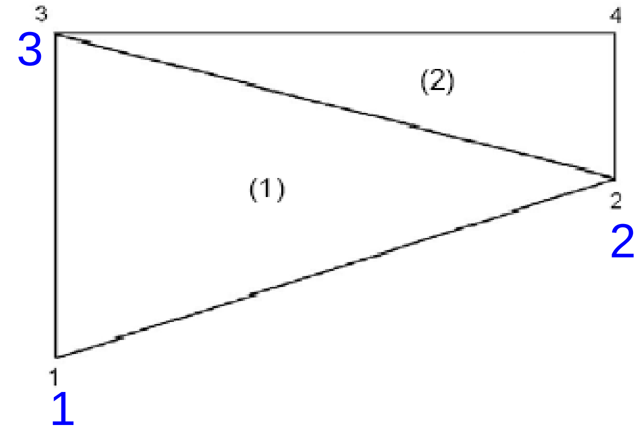


- We want to compute the temperature field using 2 linear triangular elements.



Example problem : Stiffness, element 1

- Using expressions derived before for triangular elements, we can write:



$$\underbrace{\left[B^e\right]}_{\substack{\text{constant} \\ \text{matrix}}} = \begin{bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} & \dfrac{\partial N_3^e}{\partial x} \\ \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_2^e}{\partial y} & \dfrac{\partial N_3^e}{\partial y} \end{bmatrix} = \frac{1}{2A^e}\begin{bmatrix} y_2^e - y_3^e & y_3^e - y_1^1 & y_1^e - y_2^e \\ x_3^e - x_2^e & x_1^e - x_3^e & x_2^e - x_1^e \end{bmatrix}$$

$$2A^e = \left(x_2^e y_3^e - x_3^e y_2^e\right) - \left(x_1^e y_3^e - x_3^e y_1^e\right) + \left(x_1^e y_2^e - x_2^e y_1^e\right)$$

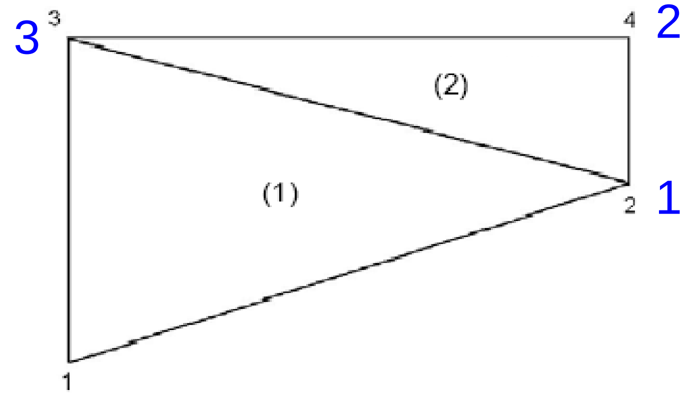
$$\left[K^e\right] = \int_{\Omega^e} \left[B^e\right]^T [D^e] \left[B^e\right] d\Omega = \left[B^e\right]^T k \left[B^e\right] A^e$$

- For *e=1*: $\left[B^1\right] = \dfrac{1}{2}\begin{bmatrix} -0.5 & 1 & -0.5 \\ -2 & 0 & 2 \end{bmatrix}$

$$K^1 = B^{1^T} B^1 k A^1 = \begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 5.3125 & -0.625 & -4.6875 \\ -0.625 & 1.25 & -0.625 \\ -4.6875 & -0.625 & 5.3125 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

Example problem : Stiffness, element 2

- Using expressions derived before for triangular elements, we can write:



$$\underbrace{B^e}_{\substack{cons\tan t \\ matrix}} = \begin{bmatrix} \dfrac{\partial N_1^e}{\partial x} & \dfrac{\partial N_2^e}{\partial x} & \dfrac{\partial N_3^e}{\partial x} \\ \dfrac{\partial N_1^e}{\partial y} & \dfrac{\partial N_2^e}{\partial y} & \dfrac{\partial N_3^e}{\partial y} \end{bmatrix} = \frac{1}{2A^e} \begin{bmatrix} y_2^e - y_3^e & y_3^e - y_1^1 & y_1^e - y_2^e \\ x_3^e - x_2^e & x_1^e - x_3^e & x_2^e - x_1^e \end{bmatrix}$$

$$2A^e = \left(x_2^e y_3^e - x_3^e y_2^e\right) - \left(x_1^e y_3^e - x_3^e y_1^e\right) + \left(x_1^e y_2^e - x_2^e y_1^e\right)$$

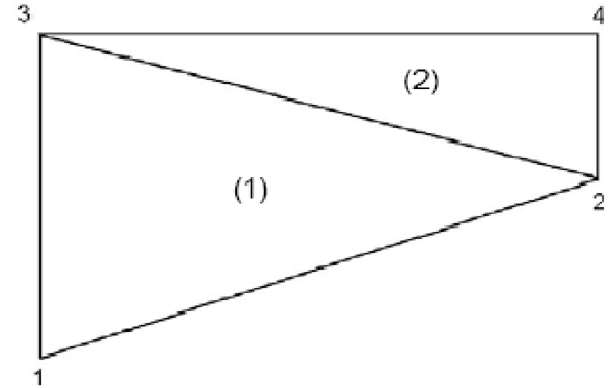
$$K^e = \int_{\Omega^e} B^{e^T} [D^e] B^e d\Omega = B^{e^T} B k A^e$$

- For *e=2*: $B^2 = \begin{bmatrix} 0 & 0.5 & -0.5 \\ -2 & 2 & 0 \end{bmatrix}$

$$\left[K^2\right] = \left[B^2\right]^T \left[B^2\right] k A^2 = \begin{array}{c} \begin{array}{ccc} 2 & 4 & 3 \end{array} \\ \begin{bmatrix} 10 & -10 & 0 \\ -10 & 10.625 & -0.625 \\ 0 & -0.625 & 0.625 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 4 \\ 3 \end{array}$$

Example problem : Assembled stiffness

- Assembling the 2 stiffness matrices

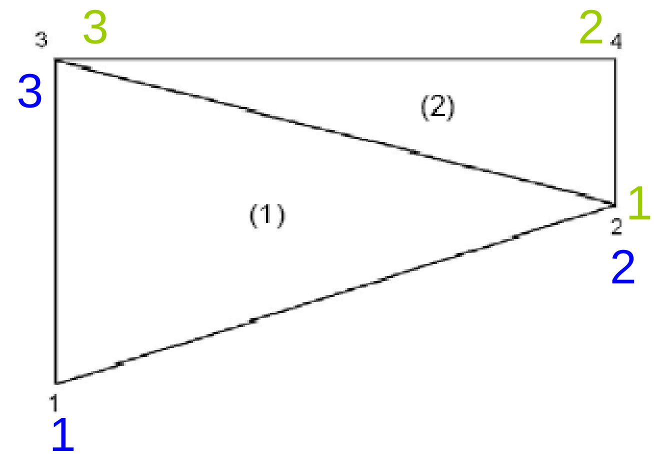


$$\left[K\right]=\left[L^1\right]^T\left[K^1\right]\left[L^1\right]+\left[L^2\right]^T\left[K^2\right]\left[L^2\right]=\begin{array}{cc}\begin{bmatrix}5.3125 & -0.625 & -4.6875 & 0\\ -0.625 & 11.25 & -0.625 & -10\\ -4.6875 & -0.625 & 5.9375 & -0.625\\ 0 & -10 & -0.625 & 10.625\end{bmatrix} & \begin{matrix}1\\2\\3\\4\end{matrix}\end{array}$$

Example problem : Load calculations – e=1,2

- The shape functions for 3-node triangular elements are

$$\left[N^e\right]=\left[N_1^e \quad N_2^e \quad N_3^e\right]$$



$$N_1^e=\frac{1}{2A^e}(x_2^e y_3^e - x_3^e y_2^e + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y)$$

$$N_2^e=\frac{1}{2A^e}(x_3^e y_1^e - x_1^e y_3^e + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y)$$

$$N_3^e=\frac{1}{2A^e}(x_1^e y_2^e - x_2^e y_1^e + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y)$$

$$2A^e=\left(x_2^e y_3^e - x_3^e y_2^e\right)-\left(x_1^e y_3^e - x_3^e y_1^e\right)+\left(x_1^e y_2^e - x_2^e y_1^e\right)$$

$$\left\{f_\Omega^e\right\}=f\int_{\Omega^e}\left\{N^e\right\}^T d\Omega=\frac{fA^e}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

This expression can be proved for constant f!

- For *e=1*:

$$\left\{f^1\right\}=\frac{fA^1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}=\frac{6\times 1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{array}{c}\begin{bmatrix}2\\2\\2\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\end{array}$$

- For *e=2*:

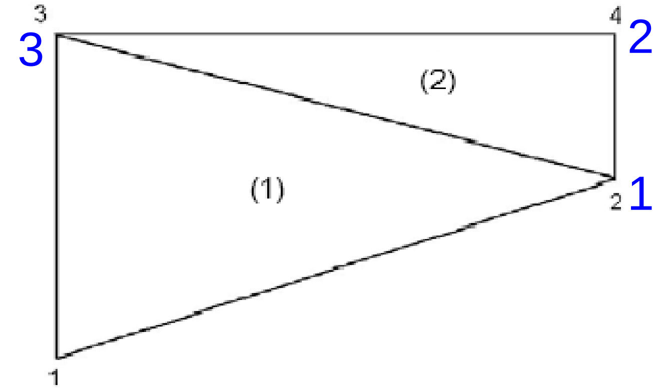
$$\left\{f^2\right\}=\frac{fA^2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}=\frac{6\times 0.5}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{array}{c}\begin{bmatrix}1\\1\\1\end{bmatrix}\end{array}\begin{array}{c}2\\4\\3\end{array}$$

$$\left\{f\right\}=\left[L^1\right]^T\left\{f^1\right\}+\left[L^2\right]^T\left\{f^2\right\}=\begin{bmatrix}2\\2+1\\2+1\\1\end{bmatrix}=\begin{array}{c}\begin{bmatrix}2\\3\\3\\1\end{bmatrix}\end{array}\begin{array}{c}1\\2\\3\\4\end{array}$$

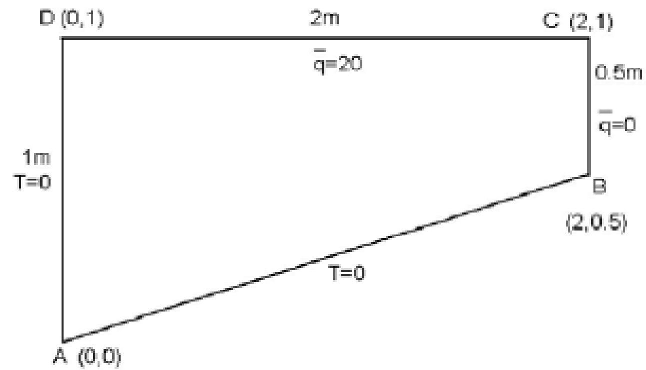
Example problem: Applying natural BCs

- For elements with segments on Γ_q, we need to compute

$$\left\{ f_\Gamma^e \right\} = -\int_{\Gamma_q^e} \left[N^e \right]^T \bar{q} \, d\Gamma$$



- We only need to worry about $e=2$



$$\left[N^2 \right]^T \Big|_{y=1} = \begin{bmatrix} \frac{1}{2A^2}(x_2^2 y_3^2 - x_3^2 y_2^2 + (y_2^2 - y_3^2)x + (x_3^2 - x_2^2)y) \\ \frac{1}{2A^2}(x_3^2 y_1^2 - x_1^2 y_3^2 + (y_3^2 - y_1^2)x + (x_1^2 - x_3^2)y) \\ \frac{1}{2A^2}(x_1^2 y_2^2 - x_2^2 y_1^2 + (y_1^2 - y_2^2)x + (x_2^2 - x_1^2)y) \end{bmatrix}_{y=1} = \begin{bmatrix} 0 \\ 0.5x \\ -0.5x+1 \end{bmatrix}$$

- Thus: $$\left\{ f_\Gamma^e \right\} = -\int_0^2 \begin{bmatrix} 0 \\ 0.5x \\ -0.5x+1 \end{bmatrix} 20 dx = \begin{array}{cc} \begin{bmatrix} 0 \\ -20 \\ -20 \end{bmatrix} & \begin{array}{c} 2 \\ 4 \\ 3 \end{array} \end{array}$$

Example problem: Final system of equations

- Assembling the load vector gives the following:

$$\{f\}=\{f_\Gamma\}+\{f_\Omega\}+\{r\}=\begin{bmatrix}2\\3\\-17\\-19\end{bmatrix}+\begin{bmatrix}r_1\\r_2\\r_3\\0\end{bmatrix}=\begin{bmatrix}r_1+2\\r_2+3\\r_3-17\\-19\end{bmatrix}$$

$r_1, r_2, r_3,$ are unknown heat fluxes at the nodes with prescribed temperature (short of reaction forces!)

- The final system of equations is:

$$\left[\begin{array}{ccc|c}5.3125 & -0.625 & -4.6875 & 0\\ -0.625 & 11.25 & -0.625 & -10\\ -4.6875 & -0.625 & 5.9375 & -0.625\\ \hline 0 & -10 & -0.625 & 10.625\end{array}\right]\left[\begin{array}{c}0\\0\\0\\ \hline T_4\end{array}\right]=\left[\begin{array}{c}r_1+2\\r_2+3\\r_3-17\\ \hline -19\end{array}\right]$$

- Partitioning this system gives: $T_4=-\dfrac{19}{10.625}=-1.788$
- `Reaction fluxes': $\begin{bmatrix}5.3125 & -0.625 & -4.6875 & 0\\ -0.625 & 11.25 & -0.625 & -10\\ -4.6875 & -0.625 & 5.9375 & -0.625\end{bmatrix}\begin{bmatrix}0\\0\\0\\-1.788\end{bmatrix}=\begin{bmatrix}r_1+2\\r_2+3\\r_3-17\end{bmatrix}\Rightarrow\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix}=\begin{bmatrix}-2\\14.88\\18.1175\end{bmatrix}$

Example problem: Postprocessing

- The DOF in each element (nodal temperatures) are now given as:

$$\left\{d^1\right\} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \left\{d^2\right\} = \begin{bmatrix} 0 \\ -1.788 \\ 0 \end{bmatrix}$$

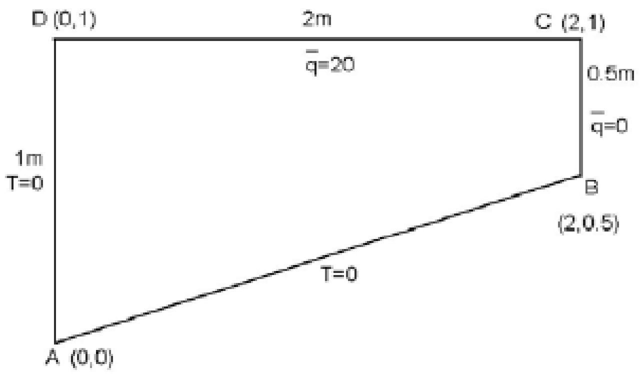
- The flux vectors can then be computed for each element as:

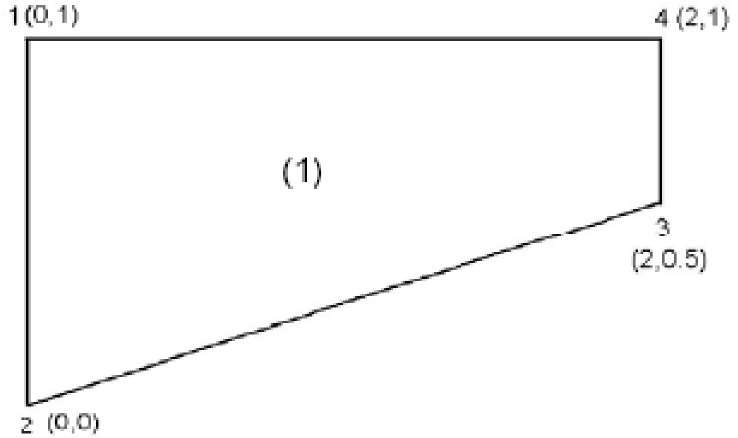
$$\left\{q^1\right\} \equiv \begin{bmatrix} q_x^1 \\ q_y^1 \end{bmatrix} = -k\left[B^1\right]\left\{d^1\right\} = -5\frac{1}{2}\begin{bmatrix} -0.5 & 1 & -0.5 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left\{q^2\right\} \equiv \begin{bmatrix} q_x^2 \\ q_y^2 \end{bmatrix} = -k\left[B^2\right]\left\{d^2\right\} = -5\begin{bmatrix} 0 & 0.5 & -0.5 \\ -2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ -1.788 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.47 \\ 17.88 \end{bmatrix}$$

Example problem using 4-node quadrilateral elements

- We repeat the calculations of the earlier example using one 4-node quadrilateral element. We use again `matrix notation'.





- The element coordinates are:

$$\begin{bmatrix} x^e & y^e \end{bmatrix} = \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 0.5 \\ 2 & 1 \end{bmatrix}$$

Example problem using 4-node quadrilateral elements

- The basis functions in natural coordinates are:

$$\widehat{N}_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta)$$
$$\widehat{N}_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$$
$$\widehat{N}_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$$
$$\widehat{N}_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$

- We need to compute the B^e matrix relating derivatives wrt x and y and nodal values.

$$\begin{bmatrix} \frac{\partial T^e}{\partial x} \\ \frac{\partial T^e}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \widehat{N}_1^e}{\partial x} & \frac{\partial \widehat{N}_2^e}{\partial x} & \frac{\partial \widehat{N}_3^e}{\partial x} & \frac{\partial \widehat{N}_4^e}{\partial x} \\ \frac{\partial \widehat{N}_1^e}{\partial y} & \frac{\partial \widehat{N}_2^e}{\partial y} & \frac{\partial \widehat{N}_3^e}{\partial y} & \frac{\partial \widehat{N}_4^e}{\partial y} \end{bmatrix}}_{[B^e]} \begin{bmatrix} T_1^e \\ T_2^e \\ T_3^e \\ T_4^e \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial T^e}{\partial x} \\ \frac{\partial T^e}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \widehat{N}_1^e}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \widehat{N}_1^e}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial \widehat{N}_2^e}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \widehat{N}_2^e}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial \widehat{N}_3^e}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \widehat{N}_3^e}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial \widehat{N}_4^e}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \widehat{N}_4^e}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial \widehat{N}_1^e}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \widehat{N}_1^e}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial \widehat{N}_2^e}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \widehat{N}_2^e}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial \widehat{N}_3^e}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \widehat{N}_3^e}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial \widehat{N}_4^e}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \widehat{N}_4^e}{\partial \eta}\frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} T_1^e \\ T_2^e \\ T_3^e \\ T_4^e \end{bmatrix}$$

Example problem using 4-node quadrilateral elements

$$\left[B^e\right]=\begin{bmatrix} \frac{\partial \widehat{N}_1^e}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \widehat{N}_1^e}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial \widehat{N}_2^e}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \widehat{N}_2^e}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial \widehat{N}_3^e}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \widehat{N}_3^e}{\partial \eta}\frac{\partial \eta}{\partial x} & \frac{\partial \widehat{N}_4^e}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial \widehat{N}_4^e}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial \widehat{N}_1^e}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \widehat{N}_1^e}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial \widehat{N}_2^e}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \widehat{N}_2^e}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial \widehat{N}_3^e}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \widehat{N}_3^e}{\partial \eta}\frac{\partial \eta}{\partial y} & \frac{\partial \widehat{N}_4^e}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial \widehat{N}_4^e}{\partial \eta}\frac{\partial \eta}{\partial y} \end{bmatrix}$$

- We can write this matrix as:

$$\left[B^e\right]=\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}\begin{bmatrix} \frac{\partial \widehat{N}_1^e}{\partial \xi} & \frac{\partial \widehat{N}_2^e}{\partial \xi} & \frac{\partial \widehat{N}_3^e}{\partial \xi} & \frac{\partial \widehat{N}_4^e}{\partial \xi} \\ \frac{\partial \widehat{N}_1^e}{\partial \eta} & \frac{\partial \widehat{N}_2^e}{\partial \eta} & \frac{\partial \widehat{N}_3^e}{\partial \eta} & \frac{\partial \widehat{N}_4^e}{\partial \eta} \end{bmatrix}=\frac{1}{|J^e|}\begin{bmatrix} \frac{\partial y^e}{\partial \eta} & -\frac{\partial y^e}{\partial \xi} \\ -\frac{\partial x^e}{\partial \eta} & \frac{\partial x^e}{\partial \xi} \end{bmatrix}\begin{bmatrix} \frac{\partial \widehat{N}_1^e}{\partial \xi} & \frac{\partial \widehat{N}_2^e}{\partial \xi} & \frac{\partial \widehat{N}_3^e}{\partial \xi} & \frac{\partial \widehat{N}_4^e}{\partial \xi} \\ \frac{\partial \widehat{N}_1^e}{\partial \eta} & \frac{\partial \widehat{N}_2^e}{\partial \eta} & \frac{\partial \widehat{N}_3^e}{\partial \eta} & \frac{\partial \widehat{N}_4^e}{\partial \eta} \end{bmatrix}$$

This comes from an earlier derivation:

$$\begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix}=\frac{1}{|J^e|}\begin{bmatrix} \frac{\partial y^e}{\partial \eta} & -\frac{\partial x^e}{\partial \eta} \\ -\frac{\partial y^e}{\partial \xi} & \frac{\partial x^e}{\partial \xi} \end{bmatrix}\begin{Bmatrix} dx \\ dy \end{Bmatrix}$$

Example problem using 4-node quadrilateral elements

$$\left[B^e\right] = \underbrace{\frac{1}{|J^e|}\begin{bmatrix} \dfrac{\partial y^e}{\partial \eta} & -\dfrac{\partial y^e}{\partial \xi} \\ -\dfrac{\partial x^e}{\partial \eta} & \dfrac{\partial x^e}{\partial \xi} \end{bmatrix}}_{\substack{\text{Inverse of} \\ \text{the Jacobian } [J^e]}} \begin{bmatrix} \dfrac{\partial \widehat{N}_1^e}{\partial \xi} & \dfrac{\partial \widehat{N}_2^e}{\partial \xi} & \dfrac{\partial \widehat{N}_3^e}{\partial \xi} & \dfrac{\partial \widehat{N}_4^e}{\partial \xi} \\ \dfrac{\partial \widehat{N}_1^e}{\partial \eta} & \dfrac{\partial \widehat{N}_2^e}{\partial \eta} & \dfrac{\partial \widehat{N}_3^e}{\partial \eta} & \dfrac{\partial \widehat{N}_4^e}{\partial \eta} \end{bmatrix}$$

- The Jacobian matrix can be computed from:

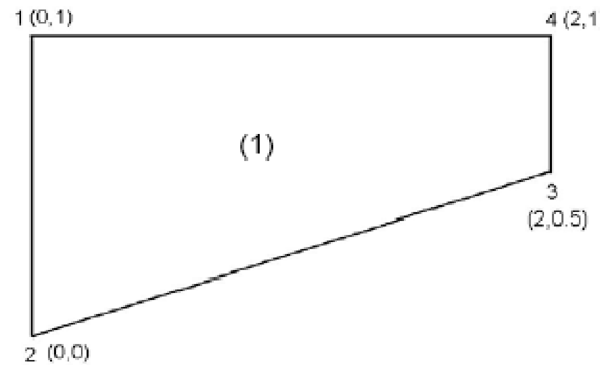
$$x^e = \sum_{j=1}^{4} x_j^e \widehat{N}_j(\xi,\eta),$$

$$y^e = \sum_{j=1}^{4} y_j^e \widehat{N}_j(\xi,\eta),$$

$$\underbrace{\left[J^e\right]}_{\substack{\text{Jacobian} \\ \text{matrix of} \\ \text{element } e}} = \begin{bmatrix} \dfrac{\partial x^e}{\partial \xi} & \dfrac{\partial y^e}{\partial \xi} \\ \dfrac{\partial x^e}{\partial \eta} & \dfrac{\partial y^e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \widehat{N}_1^e}{\partial \xi} & \dfrac{\partial \widehat{N}_2^e}{\partial \xi} & \dfrac{\partial \widehat{N}_3^e}{\partial \xi} & \dfrac{\partial \widehat{N}_4^e}{\partial \xi} \\ \dfrac{\partial \widehat{N}_1^e}{\partial \eta} & \dfrac{\partial \widehat{N}_2^e}{\partial \eta} & \dfrac{\partial \widehat{N}_3^e}{\partial \eta} & \dfrac{\partial \widehat{N}_4^e}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix}$$

Example problem using 4-node quadrilateral elements

- For element 1:



$$\left[J^1\right]=\frac{1}{4}\begin{bmatrix}\eta-1 & 1-\eta & 1+\eta & -\eta-1\\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\\ 2 & 0.5\\ 2 & 1\end{bmatrix}=\begin{bmatrix}0 & 0.125\eta-0.375\\ 1 & 0.125\xi+0.125\end{bmatrix}$$

$$\left[J^1\right]^{-1}=\begin{bmatrix}\dfrac{1+\xi}{3-\eta} & 1\\ \dfrac{8}{\eta-3} & 0\end{bmatrix}, \det J^1=-0.125\eta+0.375$$

$$\left[B^1\right]=\begin{bmatrix}\dfrac{1+\xi}{3-\eta} & 1\\ \dfrac{8}{\eta-3} & 0\end{bmatrix}\frac{1}{4}\begin{bmatrix}\eta-1 & 1-\eta & 1+\eta & -\eta-1\\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi\end{bmatrix}$$

- The element stiffness is: $\left[K^1\right]=\int_{\Omega^1}\left[B^1\right]^T\left[D^1\right]\left[B^1\right]d\Omega=\int_{-1}^{1}\int_{-1}^{1}\left[B^1\right]^T\underbrace{\left[D^1\right]}_{k[I]}\left[B^1\right]|J^1|\,d\xi d\eta\Rightarrow$

$$\left[K^1\right]=\sum_{i=1}^{N_{int}=2}\sum_{j=1}^{N_{int}=2}k\left(\left[B^1\right]^T\left[B^1\right]|J^1|\right)_{\xi=\xi_i,\eta=\eta_j}W_iW_j,\ (\xi_i,\eta_j)=(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}),(W_i,W_j)=(1,1)$$

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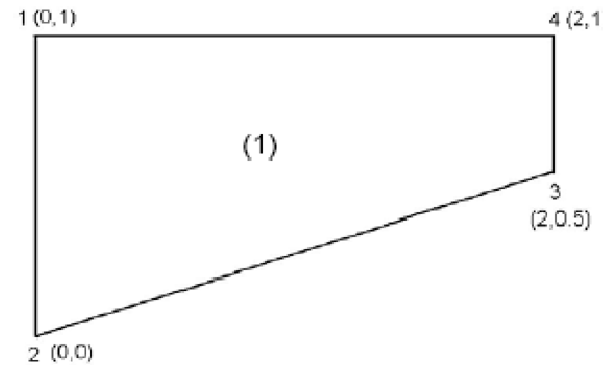
- The element stiffness is finally given as:



$$\left[K^1\right]=\begin{bmatrix} 4.76 & -3.51 & -2.98 & 1.73 \\ -3.51 & 4.13 & 1.73 & -2.36 \\ -2.98 & 1.73 & 6.54 & -5.29 \\ 1.73 & -2.36 & -5.29 & 5.91 \end{bmatrix}$$

- The nodal force is (Gauss integration not shown):

$$\left\{f_\Omega^e\right\}=\int_{\Omega^e} f\left[N^e\right]^T d\Omega=\int_{-1}^{1}\int_{-1}^{1} f\left[N^e\right]^T |J^1|\, d\xi d\eta=\int_{-1}^{1}\int_{-1}^{1} 6\begin{bmatrix}\widehat{N}_1^e\\ \widehat{N}_2^e\\ \widehat{N}_3^e\\ \widehat{N}_4^e\end{bmatrix}(-0.125\eta+0.375)d\xi d\eta=\begin{bmatrix}2.5\\2.5\\2\\2\end{bmatrix}$$

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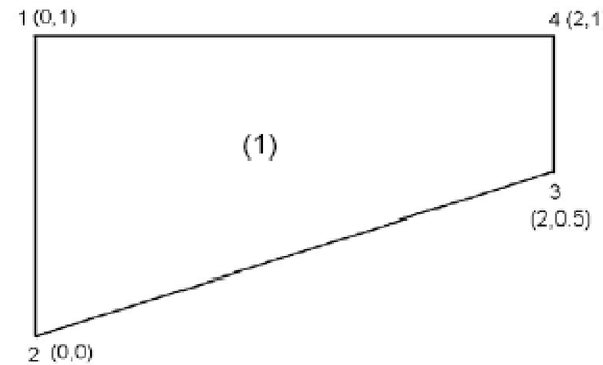
- We now need to compute the contribution to element load from the natural boundary conditions on the element *e=1* at $\xi=-1$.



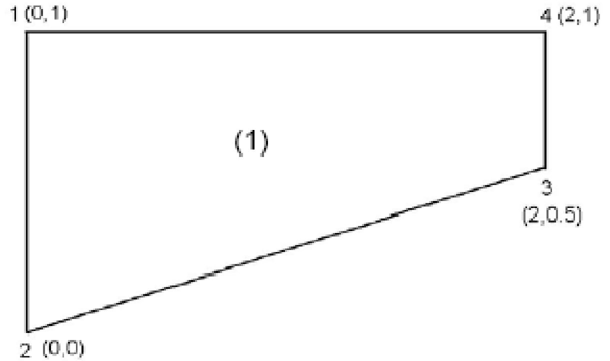
$$\left\{ f_q^e \right\} = -\int_{\Omega^e} \bar{q}\left[N^e\right]^T d\Gamma = -\int_{-1}^{1} \bar{q}\left(\left[N^e\right]^T |J^1|\right)_{\xi=-1} d\eta = -\int_{-1}^{1} 20 \begin{bmatrix} \frac{1}{2}(1-\eta) \\ 0 \\ 0 \\ \frac{1}{2}(1+\eta) \end{bmatrix} \underbrace{\frac{2-0}{2}}_{|J^1|} d\eta = \begin{bmatrix} -20 \\ 0 \\ 0 \\ -20 \end{bmatrix}$$

- Assembly of all the nodal contributions gives:

$$\{f\} = \{f_\Omega\} + \{f_\Gamma\} + \{r\} = \begin{bmatrix} r_1 - 17.5 \\ r_2 + 2.5 \\ r_3 + 2 \\ -18 \end{bmatrix}$$

Example problem using 4-node quadrilateral elements

- We finally obtain:



$$\left[\begin{array}{ccc|c} 4.76 & -3.51 & -2.98 & 1.73 \\ -3.51 & 4.13 & 1.73 & -2.36 \\ -2.98 & 1.73 & 6.54 & -5.29 \\ \hline 1.73 & -2.36 & -5.29 & 5.91 \end{array}\right]\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline T_4 \end{array}\right] = \left[\begin{array}{c} r_1 - 17.5 \\ r_2 + 2.5 \\ r_3 + 2 \\ \hline -18 \end{array}\right]$$

which gives $T_4 = -3.04.$

- The nodal DOFs are: $\left\{d^1\right\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.04 \end{bmatrix}$

Example problem using 4-node quadrilateral elements

- The flux components at each Gauss point are:

$$\left\{q^1(\xi,\eta)\right\} = \begin{bmatrix} -k\dfrac{\partial T^e}{\partial x} \\ -k\dfrac{\partial T^e}{\partial y} \end{bmatrix} = -k\left[B^e\right]\begin{bmatrix} T_1^e \\ T_2^e \\ T_3^e \\ T_4^e \end{bmatrix} = -k\begin{bmatrix} \dfrac{1+\xi}{3-\eta} & 1 \\ \dfrac{8}{\eta-3} & 0 \end{bmatrix}\frac{1}{4}\begin{bmatrix} \eta-1 & 1-\eta & 1+\eta & -\eta-1 \\ \xi-1 & -\xi-1 & 1+\xi & 1-\xi \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.04 \end{bmatrix} \Rightarrow$$

$$\left\{q^1\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\right\} = \begin{bmatrix} 0.8979 \\ 3.5916 \end{bmatrix}$$

$$\left\{q^1\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right\} = \begin{bmatrix} -2.2965 \\ 19.793 \end{bmatrix}$$

$$\left\{q^1\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right\} = \begin{bmatrix} 4.9482 \\ 19.793 \end{bmatrix}$$

$$\left\lfloor q^1\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)\right\rfloor = \begin{bmatrix} 5.8042 \\ 3.5916 \end{bmatrix}$$