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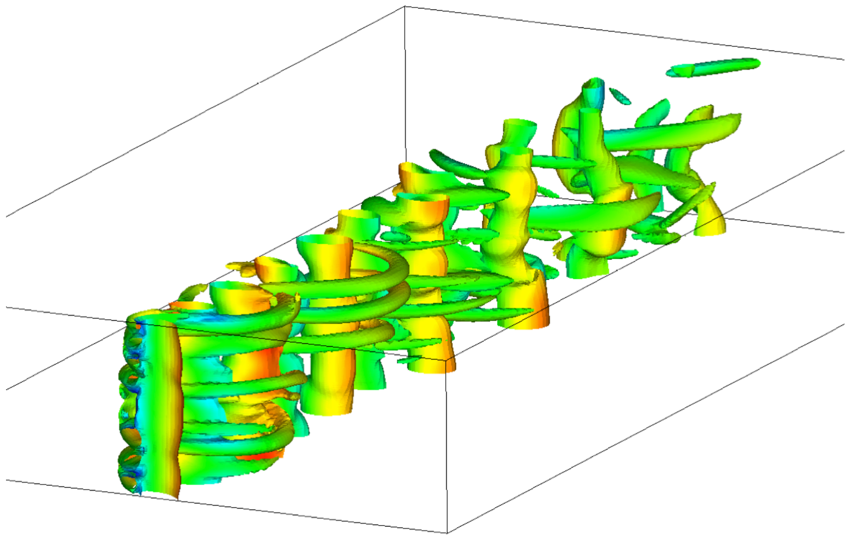

Stabilized Formulations — Core Concepts

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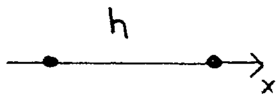
TIME-DEPENDENT ADVECTION-DIFFUSION EQUATION : 1D

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} - \nu\frac{\partial^2 \phi}{\partial x^2} = f$$

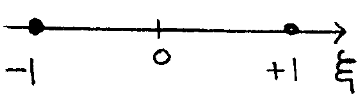
ELEMENT MATRICES :



$$\int_{\Omega^e} N_A\, u^h \frac{\partial N_B}{\partial x}\, d\Omega \qquad\qquad \int_{\Omega^e} \frac{\partial N_A}{\partial x}\, \nu\, \frac{\partial N_B}{\partial x}\, d\Omega$$

COMPARE THESE TWO MATRICES

CHANGE OF COORDINATES



$$\int_{-1}^{+1} N_A\, u^h \overset{\frac{\partial N_B}{\partial x}}{\left(\frac{2}{h}\frac{\partial N_B}{\partial \xi}\right)} \overset{d\Omega}{\left(\frac{h}{2}\, d\xi\right)} \qquad\qquad \int_{-1}^{+1} \underset{\frac{\partial N_A}{\partial x}}{\left(\frac{2}{h}\frac{\partial N_A}{\partial \xi}\right)} \nu \underset{\frac{\partial N_B}{\partial x}}{\left(\frac{2}{h}\frac{\partial N_B}{\partial \xi}\right)} \underset{d\Omega}{\left(\frac{h}{2}\, d\xi\right)}$$

$$u^h \underbrace{\int_{-1}^{+1} N_A \frac{\partial N_B}{\partial \xi}\, d\xi}_{\text{NO DIMENSION}} \qquad\qquad \frac{2}{h}\nu \underbrace{\int_{-1}^{+1} \frac{\partial N_A}{\partial \xi}\frac{\partial N_B}{\partial \xi}\, d\xi}_{\text{NO DIMENSION}}$$

COMPARE THESE TWO

$$\frac{u^h}{\frac{2}{h}\nu} = \frac{u^h h}{2\nu} \qquad \text{ELEMENT PECLET NUMBER}$$

ADVECTION-DIFFUSION EQUATION

Element Peclet Number (Cell Peclet Number) $\quad Pe_h = \dfrac{u^h h}{2\nu}$

NAVIER-STOKES EQUATIONS

Element Reynolds Number (Cell Reynolds Number) $\quad Re_h = \dfrac{u^h h}{2\nu}$

IN 2D AND 3D

$$Pe_h = \frac{\|\underset{\sim}{u}^h\| \, h}{2\nu}$$

$$Re_h = \frac{\|\underset{\sim}{u}^h\| \, h}{2\nu}$$

$\|\underset{\sim}{u}^h\|$: MAGNITUDE OF $\underset{\sim}{u}^h$

h : A MEASURE OF "ELEMENT LENGTH" (ELEMENT SCALE) IN 2D AND 3D

NUMERICAL DIFFICULTIES WHEN $Pe_h \gg 1$, $Re_h \gg 1$

OK WHEN $Pe_h \approx 1$, $Re_h \approx 1$

CONSIDER THE FOLLOWING STABILIZED FORMULATION IN 1D, FOR ADVECTION-DIFFUSION EQUATION

$$\int_\Omega w^h \frac{\partial \phi^h}{\partial t}\, d\Omega + \underbrace{\int_\Omega w^h u^h \frac{\partial \phi^h}{\partial x}\, d\Omega}_{A} + \underbrace{\int_\Omega \frac{\partial w^h}{\partial x} \nu \frac{\partial \phi^h}{\partial x}\, d\Omega}_{D}$$

STABILIZATION PARAMETER

S

$$\underbrace{+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau u^h \frac{\partial w^h}{\partial x} \underbrace{\left(\frac{\partial \phi^h}{\partial t} + u^h \frac{\partial \phi^h}{\partial x} - \nu \frac{\partial^2 \phi^h}{\partial x^2} - f \right)}_{\text{DIFFERENTIAL EQUATION}} d\Omega}_{} = \int_{\Gamma_h} w^h h^h\, d\Gamma + \int_\Omega w^h f^h\, d\Omega$$

NEW TERMS: STABILIZATION TERMS

WHAT IS THE DIMENSION OF τ ?

NOTE: THE NEW FORMULATION IS STILL CONSISTENT.

COMPARE D AND S

$$\to \int_{\Omega^e} \tau u^h \frac{\partial w^h}{\partial x} u^h \frac{\partial \phi^h}{\partial x}\, d\Omega$$

$$= \int_{\Omega^e} \frac{\partial w^h}{\partial x} \left(\tau (u^h)^2 \right) \frac{\partial \phi^h}{\partial x}\, d\Omega$$

$$\to \int_{\Omega^e} \frac{\partial w^h}{\partial x} \nu \frac{\partial \phi^h}{\partial x}\, d\Omega$$

$\tau (u^h)^2$ HAS THE SAME DIMENSION AS ν

CALL $\tau (u^h)^2 = \tilde{\nu}$

IF WE COMPARE A TO D, WE GET

$$Pe_h = \frac{u^h h}{2\nu}$$

IF WE COMPARE A TO S, WE GET

$$\widetilde{Pe}_h = \frac{u^h h}{2\tilde{\nu}} = \frac{u^h h}{2\tau(u^h)^2} = \frac{h}{2u^h}\frac{1}{\tau}$$

ASSUME THAT $Pe_h >> 1$ ($\Rightarrow$ NUMERICAL DIFFICULTY IF WE DO NOT DO SOMETHING)

THEN WE WANT $\widetilde{Pe}_h \approx 1$

$$\Rightarrow \quad \frac{h}{2u^h}\frac{1}{\tau} \approx 1 \quad \Rightarrow \quad \tau = \underbrace{\frac{h}{2u^h}}_{\text{ONE OF SEVERAL SELECTIONS}}$$

STABILIZED FORMULATION FOR ADVECTION-DIFFUSION EQUATION IN 2D AND 3D :

$$\int_\Omega w^h \frac{\partial \phi^h}{\partial t}\, d\Omega + \underbrace{\int_\Omega w^h (\underset{\sim}{u}^h\cdot\nabla)\phi^h\, d\Omega}_{A} + \underbrace{\int_\Omega \underset{\sim}{\nabla} w^h \cdot \nu\, \underset{\sim}{\nabla}\phi^h\, d\Omega}_{D}$$

$$\underbrace{+\sum_{e=1}^{n_{el}} \int \tau(\underset{\sim}{u}^h\cdot\underset{\sim}{\nabla}) w^h \underbrace{\left(\frac{\partial \phi^h}{\partial t} + (\underset{\sim}{u}^h\cdot\underset{\sim}{\nabla})\phi^h - \nu\nabla^2\phi^h - f\right)}_{\text{DIFFERENTIAL EQUATION}} d\Omega}_{S} = \int_{\Gamma_h} w^h h^h\, d\Gamma + \int_\Omega w^h f^h\, d\Omega$$

NEW TERMS : STABILIZATION TERMS

COMPARE D AND S

$$D \rightarrow \int_{\Omega^e} \underset{\sim}{\nabla} w^h \cdot \nu\, \underset{\sim}{\nabla}\phi^h\, d\Omega$$

$$S \rightarrow \int_{\Omega^e} \tau(\underset{\sim}{u}^h\cdot\underset{\sim}{\nabla}) w^h (\underset{\sim}{u}^h\cdot\underset{\sim}{\nabla})\phi^h\, d\Omega$$

$$= \int_{\Omega^e} \underset{\sim}{\nabla} w^h \cdot \underbrace{\left(\tau\, \underset{\sim}{u}^h \underset{\sim}{u}^h\right)}_{\underset{\sim}{\tilde{\nu}}} \cdot \underset{\sim}{\nabla}\phi^h\, d\Omega$$

$\underset{\sim}{\tilde{\nu}}$: SIMILAR TO WHAT WE HAD IN 1D, <u>BUT</u> NOW $\underset{\sim}{\tilde{\nu}}$ IS A MATRIX

CONSIDER THE DIRECTION ALONG THE STREAMLINE, UNIT VECTOR :

$$\underset{\sim}{s} = \frac{\underset{\sim}{u}^h}{\|\underset{\sim}{u}^h\|}$$

ALONG THE STREAMLINE:

$$\underset{\sim}{s} \cdot \left(\tau\, \underline{u}^h \underline{u}^h \right) \cdot \underset{\sim}{s}$$

$$= \tau \underbrace{\left(\underset{\sim}{s} \cdot \underline{u}^h\right)}_{\|\underset{\sim}{u}^h\|} \underbrace{\left(\underline{u}^h \cdot \underset{\sim}{s}\right)}_{\|\underline{u}^h\|}$$

$$= \tau \, \|\underline{u}^h\|^2$$

WE WANT

$$\widetilde{Pe}_h \approx 1$$

$$\widetilde{Pe}_h = \frac{h}{2\|\underset{\sim}{u}^h\|} \frac{1}{\tau} \quad \Rightarrow \quad \underbrace{\tau = \frac{h}{2\|\underset{\sim}{u}^h\|}}_{\text{ONE OF SEVERAL SELECTIONS.}}$$

CONSIDER A DIRECTION $\underset{\sim}{s}_\perp$ PERPENDICULAR TO $\underset{\sim}{s}$

$$\left(\Rightarrow \quad \underset{\sim}{s}_\perp \cdot \underset{\sim}{s} = 0 \;, \quad \underset{\sim}{s}_\perp \cdot \underline{u}^h = 0 \right)$$

$$\underset{\sim}{s}_\perp \cdot \left(\tau\, \underline{u}^h \underline{u}^h \right) \cdot \underset{\sim}{s}_\perp = \tau \underbrace{\left(\underset{\sim}{s}_\perp \cdot \underline{u}^h\right)}_{0} \underbrace{\left(\underline{u}^h \cdot \underset{\sim}{s}_\perp\right)}_{0} = 0$$

$\Rightarrow$ $\boxed{\text{THE STABILIZATION IS CALLED STREAMLINE-UPWIND / PETROV-GALERKIN (SUPG) FORMULATION}}$

SUPG STABILIZATION FOR THE NAVIER-STOKES EQUATIONS OF INCOMPRESSIBLE FLOWS

$$\int_{\Omega} \underset{\sim}{w}^h \cdot \rho\left(\frac{\partial \underset{\sim}{u}^h}{\partial t} + (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla})\underset{\sim}{u}^h - \underset{\sim}{f}\right) d\Omega + \int_{\Omega} \underset{\sim}{\varepsilon}(\underset{\sim}{w}^h) : \underset{\sim}{\sigma}(p^h, \underset{\sim}{u}^h)\, d\Omega$$

$$+ \int_{\Omega} q^h (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}^h)\, d\Omega + \underbrace{\sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{SUPG} (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla}) \underset{\sim}{w}^h \cdot \underbrace{\left[\rho\left(\frac{\partial \underset{\sim}{u}^h}{\partial t} + (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla})\underset{\sim}{u}^h - \underset{\sim}{f}\right) - \underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma}\right]}_{\text{MOMENTUM EQUATION}} d\Omega}_{\text{SUPG STABILIZATION TERM}}$$

$$= \int_{\Gamma_h} \underset{\sim}{w}^h \cdot \underset{\sim}{h}^h\, d\Gamma$$

NOTE THAT $\underset{\sim}{w}^h$ IS THE TEST FUNCTION CORRESPONDING TO $\underset{\sim}{u}^h$
q^h " " " " " p^h

REMARKS:

I) WITHOUT ADDITIONAL STABILIZATION, THERE ARE RESTRICTIONS ON THE COMBINATION OF INTERPOLATION FUNCTIONS THAT CAN BE USED FOR $\underset{\sim}{u}^h$ AND p^h ($\underset{\sim}{w}^h$ AND q^h).

FOR EXAMPLE IF WE USE LINEAR FUNCTIONS FOR BOTH $\underset{\sim}{u}^h$ AND p^h, PRESSURE OSCILLATIONS WILL BE GENERATED DUE TO NUMERICAL INSTABILITY

II) EQUATION SYSTEM:

$$\begin{array}{cc} & \overbrace{\begin{array}{cc}\quad\Delta \underset{\sim}{u}^h \quad\quad & \quad \Delta p^h\end{array}}^{\text{UNKNOWNS}} \\ \text{EQUATIONS}\left\{\begin{array}{c}\underset{\sim}{w}^h \\ \\ q^h\end{array}\right. & \left[\begin{array}{c|c} (\text{NON-ZERO}) & (\text{NON-ZERO}) \\ \hline (\text{NON-ZERO}) & \underset{\sim}{0}\end{array}\right] \end{array} \begin{Bmatrix}\Delta\underset{\sim}{u} \\ \Delta \underset{\sim}{p}\end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{F}^* \\ \underset{\sim}{E}^*\end{Bmatrix}$$

nodal values for $\Delta \underset{\sim}{u}^h$

nodal values for Δp^h

PROBLEM !!

BECAUSE OF REMARKS I) AND II),

ADDITIONAL STABILIZATION IS NEEDED.

PRESSURE STABILIZATION ...

PRESSURE-STABILIZING / PETROV-GALERKIN (PSPG) FORMULATION FOR NAVIER-STOKES EQUATIONS OF INCOMRESSIBLE FLOWS

$$\int_\Omega \underset{\sim}{w}^h \cdot \rho\left(\frac{\partial \underset{\sim}{u}^h}{\partial t} + (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla})\underset{\sim}{u}^h - \underset{\sim}{f}\right) d\Omega + \int_\Omega \underset{\sim}{\varepsilon}(\underset{\sim}{w}^h) : \underset{\sim}{\sigma}(p^h, \underset{\sim}{u}^h)\, d\Omega$$

$$+ \int_\Omega q^h (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}^h)\, d\Omega$$

$$+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{SUPG} (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla}) \underset{\sim}{w}^h \cdot \underbrace{\left[\rho\left(\frac{\partial \underset{\sim}{u}^h}{\partial t} + (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla})\underset{\sim}{u}^h - \underset{\sim}{f}\right) - \underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma}\right]}_{\text{MOMENTUM EQUATION}} d\Omega$$

$$\underbrace{+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{PSPG} \frac{1}{\rho} \underset{\sim}{\nabla} q^h \cdot \overbrace{\left[\rho\left(\frac{\partial \underset{\sim}{u}^h}{\partial t} + (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla})\underset{\sim}{u}^h - \underset{\sim}{f}\right) - \underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma}\right]} d\Omega}_{\text{PSPG STABILIZATION TERMS}}$$

$$= \int_{\Gamma_h} \underset{\sim}{w}^h \cdot \underset{\sim}{h}^h\, d\Gamma$$

WHAT IS THE DIMENSION OF τ_{PSPG} ?

THE SAME AS τ_{SUPG}

$$\tau_{PSPG} = \underbrace{\frac{h}{2\|\underset{\sim}{u}^h\|}}_{\text{ONE OF SEVERAL SELECTIONS}}$$

INSPECT THE FOLLOWING PSPG TERM:

$$\int_{\Omega^e} \tau_{PSPG} \frac{1}{\rho} \nabla q^h \cdot \underbrace{\left[- \nabla \cdot \sigma \right]}_{\nabla p^h - 2\mu \nabla \cdot \varepsilon(u^h)} \, d\Omega$$

NOW INSPECT THE PSPG TERM:

$$\int_{\Omega^e} \underbrace{\tau_{PSPG} \frac{1}{\rho} \nabla q^h \cdot \nabla p^h}_{\text{SYMMETRIC}} \, d\Omega$$

CONSEQUENCES OF PSPG STABILIZATION:

I) WE CAN USE EQUAL-ORDER INTERPOLATIONS FOR u^h AND p^h (INCLUDING LINEAR u^h – LINEAR p^h)

II) THE NEW EQUATION SYSTEM:

$$\begin{array}{c|c|c|} & \Delta u^h & \Delta p^h \\ \hline w^h & \text{(NON-ZERO)} & \text{NON-ZERO} \\ \hline q^h & \text{(NON-ZERO)} & \text{(NON-ZERO)} \\ \hline \end{array} \left\{ \begin{array}{c} \Delta U \\ \hline \Delta P \end{array} \right\} = \left\{ \begin{array}{c} F^{**} \\ \hline F^{**} \end{array} \right\}$$

COMES FROM

$$\int_{\Omega^e} \tau_{PSPG} \frac{1}{\rho} \nabla q^h \cdot \nabla p^h \, d\Omega$$

STABILIZATION FOR FLOWS WITH VERY HIGH REYNOLDS NUMBER

ADDITIONAL STABILIZATION TERM :

$$+ \sum_{e=1}^{n_{el}} \underbrace{\int_{\Omega^e} \tau_{LSIC} (\nabla \cdot \mathbf{w}^h) \rho (\nabla \cdot \mathbf{u}^h) \, d\Omega}_{\text{LSIC STABILIZATION TERM}}$$

WE NOTE THAT THIS IS A LEAST-SQUARES TERM BASED ON THE INCOMPRESSIBILITY CONSTRAINT.

IT IS SYMMETRIC.

WHAT IS THE DIMENSION OF τ_{LSIC} ?

$$L^2/T$$

$$\tau_{LSIC} = \underbrace{\frac{\|\mathbf{u}^h\| \, h}{2}}_{\text{ONE OF SEVERAL SELECTIONS}}$$

GALERKIN / LEAST-SQUARES (GLS) STABILIZATION

$$\int_\Omega \underset{\sim}{w}^h \cdot \rho\left(\frac{\partial \underset{\sim}{u}^h}{\partial t} + (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla})\underset{\sim}{u}^h - \underset{\sim}{f}\right) d\Omega \; + \int_\Omega \underset{\sim}{\varepsilon}(\underset{\sim}{w}^h) : \underset{\sim}{\sigma}(p^h, \underset{\sim}{u}^h)\, d\Omega$$

$$+ \int_\Omega q^h \,(\underset{\sim}{\nabla} \cdot \underset{\sim}{u}^h)\, d\Omega$$

$$+ \underbrace{\sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{LSME} \frac{1}{\rho}\left[\overbrace{\rho\left(\frac{\partial \underset{\sim}{w}^h}{\partial t} + (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla})\, \underset{\sim}{w}^h\right) - \underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma}(q^h, \underset{\sim}{w}^h)}^{\text{VARIATION OF THE MOMENTUM EQUATION}}\right] \cdot \left[\underbrace{\rho\left(\frac{\partial \underset{\sim}{u}^h}{\partial t} + (\underset{\sim}{u}^h \cdot \underset{\sim}{\nabla})\underset{\sim}{u}^h - \underset{\sim}{f}\right) - \underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma}(p^h, \underset{\sim}{u}^h)}_{\text{MOMENTUM EQUATION}}\right] d\Omega}_{\text{THIS IS A LEAST-SQUARES TERM BASED ON THE MOMENTUM EQUATION}}$$

$$+ \sum_{e=1}^{n_{el}} \int_{\Omega^e} \tau_{LSIC}\, (\underset{\sim}{\nabla} \cdot \underset{\sim}{w}^h)\, \rho\, (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}^h)\, d\Omega$$

$$= \int_{\Gamma_h} \underset{\sim}{w}^h \cdot \underset{\sim}{h}^h \, d\Gamma$$

DECOMPOSING THE GLS STABILIZATION TERMS

FIRST: NOTE THAT $-\underset{\sim}{\nabla}\cdot\underset{\sim}{\sigma}(q^h, \underline{w}^h) = \underset{\sim}{\nabla} q^h - 2\mu\, \underset{\sim}{\nabla}\cdot\underset{\sim}{\varepsilon}(\underset{\sim}{w}^h)$

$$+ \int_{\Omega^e} \tau_{LSME} \frac{1}{\rho}\left[\rho\left(\frac{\partial \underline{w}^h}{\partial t}\right)\right]\cdot\left[\underline{\text{M}}\text{OMENTUM } \underline{\text{E}}\text{QUATION}\right] d\Omega$$

▶ NEW TERM
- OPTIONAL
- REQUIRES SPACE-TIME FORMULATION

$$+ \int_{\Omega^e} \tau_{LSME} \frac{1}{\rho}\left[\rho\,(\underset{\sim}{u}^h\cdot\underset{\sim}{\nabla})\,\underset{\sim}{w}^h\right]\cdot\left[\text{M.E.}\right] d\Omega$$

SUPG TERM

$$+ \int_{\Omega^e} \tau_{LSME} \frac{1}{\rho}\left[\underset{\sim}{\nabla} q^h\right]\cdot\left[\text{M.E.}\right] d\Omega$$

PSPG TERM

$$+ \int_{\Omega^e} \tau_{LSME} \frac{1}{\rho}\left[-2\mu\, \underset{\sim}{\nabla}\cdot\underset{\sim}{\varepsilon}(\underline{w}^h)\right]\cdot\left[\text{M.E.}\right] d\Omega$$

▶ NEW TERM
CAN BE NEGLECTED FOR $Re_h \gg 1$

If you would like to know more about these stabilized formulations, please see the following references.

T.E. Tezduyar, "Stabilized Finite Element Formulations for Incompressible Flow Computations", *Advances in Applied Mechanics*, **28** (1991) 1-44.

T. Tezduyar, "CFD Methods for Three-Dimensional Computation of Complex Flow Problems", *Journal of Wind Engineering and Industrial Aerodynamics*, **81** (1999) 97-116.