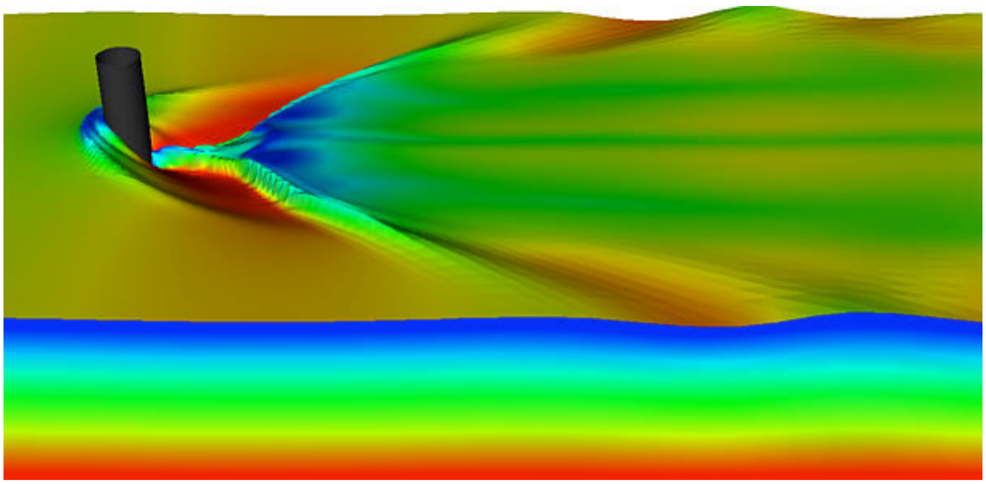
T★AFSM



Spatial Discretization for Incompressible Flows

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SPATIAL DISCRETIZATION OF THE NAVIER-STOKES EQUATIONS OF INCOMPRESSIBLE FLOWS

$$
\boxed{
\begin{aligned}
&\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \mathbf{f}\right) - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \mathbf{0} \\
&\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\
&\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_{\mathbf{g}} \\
&\mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{h} \quad \text{on } \Gamma_{\mathbf{h}} \\
&\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 \quad \text{on } \Omega
\end{aligned}
}
$$

PARTIAL DIFFERENTIAL EQUATIONS

START

SPATIAL DISCRETIZATION

$$
\boxed{
\begin{aligned}
&\mathbf{M}\dot{\mathbf{U}} + \mathbf{N}(\mathbf{U}) + \mathbf{K}\,\mathbf{U} - \mathbf{G}\mathbf{P} = \mathbf{F} \\
&\mathbf{G}^T\mathbf{U} = \mathbf{E} \\
&\mathbf{U}(0) = \mathbf{U}_0
\end{aligned}
}
$$

NONLINEAR ORDINARY DIFFERENTIAL EQUATION SYSTEM WITH CONSTRAINT

WHERE ...

UNKNOW DISCRETE VALUES OF $\underset{\sim}{u}$ AND p

$$\underset{\sim}{U} = \begin{Bmatrix} (u_1)_1 \\ (u_2)_1 \\ (u_3)_1 \\ (u_1)_2 \\ (u_2)_2 \\ (u_3)_2 \\ \cdot \\ \cdot \\ \cdot \\ (u_1)_N \\ (u_2)_N \\ (u_3)_N \end{Bmatrix} \qquad \underset{\sim}{P} = \begin{Bmatrix} (p)_1 \\ (p)_2 \\ \vdots \\ (p)_N \end{Bmatrix}$$

NODE NUMBERS

KNOWN

DISCRETE VALUES OF "FORCE"

INCLUDE

$$\underset{\sim}{F} = \begin{Bmatrix} (F_1)_1 \\ (F_2)_1 \\ (F_3)_1 \\ (F_1)_2 \\ (F_2)_2 \\ (F_3)_2 \\ \cdot \\ \cdot \\ \cdot \\ (F_1)_N \\ (F_2)_N \\ (F_3)_N \end{Bmatrix} \qquad \underset{\sim}{E} = \begin{Bmatrix} (E)_1 \\ (E)_2 \\ \vdots \\ (E)_N \end{Bmatrix} \longleftarrow \underset{\sim}{g}$$

$$\underset{\sim}{F} \longleftarrow \underset{\sim}{g}, \underset{\sim}{h}, \underset{\sim}{f}, \dot{\underset{\sim}{g}}$$

WHERE . . .

$\underset{\sim}{M}$: MASS MATRIX

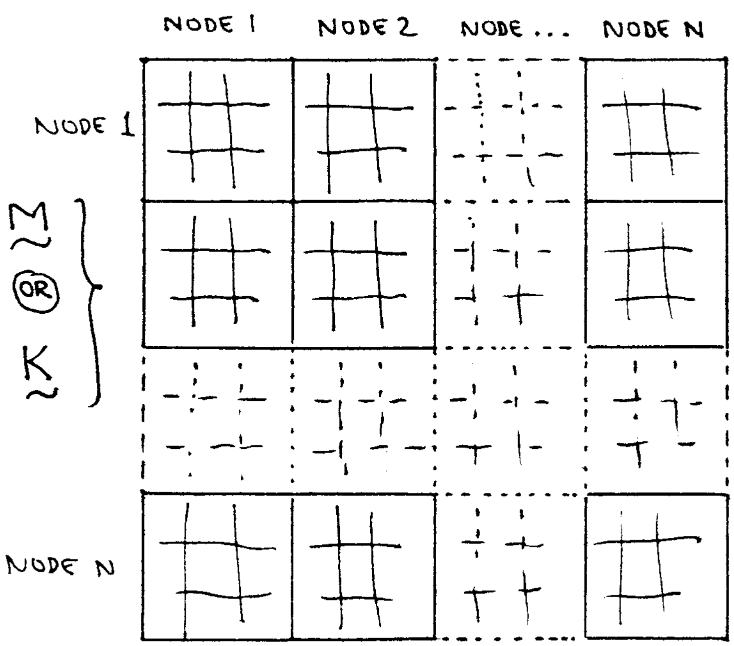
$\underset{\sim}{N}(\underset{\sim}{U})$: NONLINEAR FUNCTION (COMING FROM ADVECTIVE TERMS)

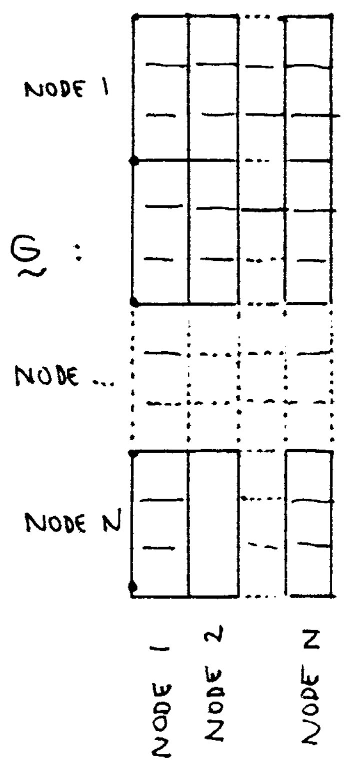
$\underset{\sim}{K}$: "STIFFNESS" MATRIX (COMING FROM VISCOUS TERMS)

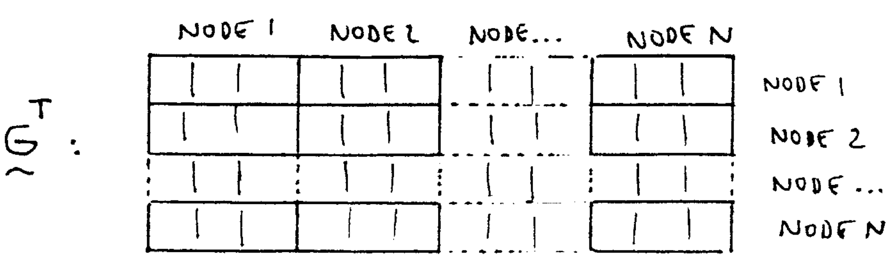
$\underset{\sim}{G}$: DISCRETE GRADIENT OPERATOR

$\underset{\sim}{G}^T$: DISCRETE DIVERGENCE OPERATOR

(NOTE : $\underset{\sim}{G}^T = \underset{\sim}{G}$ - TRANSPOSE)







$$\underset{\sim}{N}(\underset{\sim}{U}) = \begin{Bmatrix} (N_1)_1 \\ (N_2)_1 \\ (N_3)_1 \\ (N_1)_2 \\ (N_2)_2 \\ (N_3)_2 \\ \cdot \\ \cdot \\ \cdot \\ (N_1)_N \\ (N_2)_N \\ (N_3)_N \end{Bmatrix}$$

NODE NUMBERS

COMES FROM

$$\rho\,(\underbrace{\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\,\underset{\sim}{u}}_{\text{NONLINEARITY}}$$

FOR NEWTON-RAPHSON ITERATIONS

WE WILL NEED:

$$\frac{\partial \underset{\sim}{N}}{\partial \underset{\sim}{U}}$$: TANGENT "STIFFNESS" MATRIX (BASED ON THE NONLINEAR FUNCTION $\underset{\sim}{N}(\underset{\sim}{U})$)

SIMILAR TO

$\underset{\sim}{M}$ AND $\underset{\sim}{K}$

FINITE ELEMENT EQUATIONS

STEP 1:
$$\int_{\Omega} \underset{\sim}{w} \cdot \left(\rho \left(\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}) \underset{\sim}{u} - \underset{\sim}{f} \right) - \underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma} \right) d\Omega = 0$$

$$\int_{\Omega} q \, (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) \, d\Omega = 0$$

STEP 2: INTEGRATE THIS TERM BY PART

REMEMBER : a) $\underset{\sim}{W} = \underset{\sim}{0}$ ON $\Gamma_{\underset{\sim}{g}}$

b) $\underset{\sim}{n} \cdot \underset{\sim}{\sigma} = \underset{\sim}{h}$ ON $\Gamma_{\underset{\sim}{h}}$

c) $$\int_{\Omega} \underset{\sim}{W} \cdot (\underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma}) \, d\Omega = \int_{\Gamma} \underset{\sim}{W} \cdot (\underset{\sim}{n} \cdot \underset{\sim}{\sigma}) \, d\Gamma - \int_{\Omega} \underset{\sim}{\nabla} \underset{\sim}{W} : \underset{\sim}{\sigma} \, d\Omega$$

$$\int_\Omega \boldsymbol{w}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,d\Omega = \int_{\Gamma_{\boldsymbol{g}}} \underbrace{\boldsymbol{w}}_{0}\cdot(\boldsymbol{n}\cdot\boldsymbol{\sigma})\,d\Gamma + \int_{\Gamma_{\boldsymbol{h}}} \boldsymbol{w}\cdot\underbrace{(\boldsymbol{n}\cdot\boldsymbol{\sigma})}_{\boldsymbol{h}}\,d\Gamma$$

$$- \int_\Omega \underbrace{\boldsymbol{\nabla}\boldsymbol{w} : \boldsymbol{\sigma}}_{\underbrace{\begin{pmatrix}\text{SYMMETRIC PART}\\ \text{OF } \boldsymbol{\nabla}\boldsymbol{w}\end{pmatrix} : \boldsymbol{\sigma}}_{\boldsymbol{\varepsilon}(\boldsymbol{w}) : \boldsymbol{\sigma}}}\,d\Omega$$

$$= \int_{\Gamma_{\boldsymbol{h}}} \boldsymbol{w}\cdot\boldsymbol{h}\,d\Gamma - \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{w}) : \boldsymbol{\sigma}\,d\Omega$$

$$\Rightarrow \boxed{\begin{aligned} &\int_\Omega \boldsymbol{w}\cdot\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \boldsymbol{f}\right)d\Omega + \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{w}) : \boldsymbol{\sigma}\,d\Omega \\ &= \int_{\Gamma_{\boldsymbol{h}}} \boldsymbol{w}\cdot\boldsymbol{h}\,d\Gamma \quad ; \quad \int_\Omega q\,(\boldsymbol{\nabla}\cdot\boldsymbol{u})\,d\Omega = 0 \end{aligned}}$$

BECAUSE $\boldsymbol{w}$ AND q ARE INDEPENDENT, THESE EQUATIONS CAN BE ADDED SIDE-BY-SIDE. $\Longrightarrow$

$$\boxed{\begin{aligned}&\int_{\Omega} \underset{\sim}{w} \cdot \rho\left(\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u} - \underset{\sim}{f}\right) d\Omega + \int_{\Omega} \underset{\sim}{\varepsilon}(\underset{\sim}{w}) : \underset{\sim}{\sigma}\, d\Omega \\ &+ \int_{\Omega} q\,(\underset{\sim}{\nabla}\cdot\underset{\sim}{u})\, d\Omega = \int_{\Gamma_{\underset{\sim}{h}}} \underset{\sim}{w}\cdot\underset{\sim}{h}\, d\Gamma\end{aligned}}$$

NOTE THAT $\quad \underset{\sim}{\sigma} = -p\underset{\sim}{I} + 2\mu\underset{\sim}{\varepsilon}(\underset{\sim}{u})$

$$\Rightarrow \int_{\Omega} \underset{\sim}{\varepsilon}(\underset{\sim}{w}) : \underset{\sim}{\sigma}(p, \underset{\sim}{u})\, d\Omega$$

$$= \int_{\Omega} \underbrace{\underset{\sim}{\varepsilon}(\underset{\sim}{w}) : (-p\underset{\sim}{I})}_{\underbrace{-\,tr(\underset{\sim}{\varepsilon}(\underset{\sim}{w}))\,p}_{-(\underset{\sim}{\nabla}\cdot\underset{\sim}{w})\,p}} d\Omega + \int_{\Omega} \underset{\sim}{\varepsilon}(\underset{\sim}{w}) : 2\mu\,\underset{\sim}{\varepsilon}(\underset{\sim}{u})\, d\Omega$$

ELEMENT-LEVEL VECTORS

$\underset{\sim}{m}_v$:

$$\underset{\sim}{w} \cdot \rho \frac{\partial \underset{\sim}{u}}{\partial t}$$

$$c_\alpha^i N_\alpha \underset{\sim}{e}^i \cdot \rho \frac{\partial \underset{\sim}{u}}{\partial t}$$

$$c_\alpha^i \quad \boxed{N_a \rho \frac{\partial u_i}{\partial t}} \qquad \frac{(u_i)_{n+1} - (u_i)_n}{\Delta t}$$

$\underset{\sim}{c}_v$:

$$\underset{\sim}{w} \cdot \rho (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}) \underset{\sim}{u}$$

$$c_a^i N_a \underset{\sim}{e}^i \cdot \rho (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}) \underset{\sim}{u}$$

$$c_a^i \quad \boxed{N_a \; \rho (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}) u_i}$$

$\underset{\sim}{k}_v$:

$$\underset{\sim}{\varepsilon}(\underset{\sim}{w}) : 2\mu \underset{\sim}{\varepsilon}(\underset{\sim}{u})$$

$$\underset{\sim}{\nabla} \underset{\sim}{w} : 2\mu \underset{\sim}{\varepsilon}(\underset{\sim}{u})$$

$$\underset{\sim}{\nabla}(c_a^i N_a \underset{\sim}{e}^i) : 2\mu \underset{\sim}{\varepsilon}(\underset{\sim}{u})$$

$$c_a^i \quad \boxed{(\underset{\sim}{\nabla} N_a) \underset{\sim}{e}^i : 2\mu \underset{\sim}{\varepsilon}(\underset{\sim}{u})}$$

CAN ALSO BE WRITTEN AS $\longrightarrow$

$$(\underset{\sim}{\nabla} N_a)\,\underset{\sim}{e}^i : 2\mu\,\underset{\sim}{\varepsilon}(\underset{\sim}{u})$$

$$= (\underset{\sim}{\nabla} N_a)\,\underset{\sim}{e}^i : \mu\left(\underset{\sim}{\nabla}\underset{\sim}{u} + (\underset{\sim}{\nabla}\underset{\sim}{u})^T\right)$$

$$= \mu\left((\underset{\sim}{\nabla} N_a)\,\underset{\sim}{e}^i : \underset{\sim}{\nabla}\underset{\sim}{u} + (\underset{\sim}{\nabla} N_a)\,\underset{\sim}{e}^i \cdot\cdot\, \underset{\sim}{\nabla}\underset{\sim}{u}\right)$$

$$= \boxed{\mu\left(\underset{\sim}{\nabla} N_a \cdot \underset{\sim}{\nabla} u_i + \underset{\sim}{\nabla} N_a \cdot \frac{\partial \underset{\sim}{u}}{\partial x_i}\right)}$$

$\underset{\sim}{g}_v$:

$$(\underset{\sim}{\nabla}\cdot\underset{\sim}{w})\,p$$

$$\underset{\sim}{\nabla}\cdot(c_a^i N_a \underset{\sim}{e}^i)\,p$$

$$c_a^i \qquad (\underset{\sim}{e}^i \cdot \underset{\sim}{\nabla} N_a)\,p$$

$$c_a^i \qquad \boxed{\frac{\partial N_a}{\partial x_i}\,p}$$

$\boldsymbol{g}_v^T$:

$$q\,(\boldsymbol{\nabla}\cdot\boldsymbol{u})$$

$$c_a N_a\,(\boldsymbol{\nabla}\cdot\boldsymbol{u})$$

$$c_a \quad \boxed{N_a\,(\boldsymbol{\nabla}\cdot\boldsymbol{u})}$$

NOTE :

$$\mathop{\mathsf{A}}_{e=1}^{n_{el}}$$

$$\boldsymbol{N}(\boldsymbol{U}) \longleftarrow (\boldsymbol{c}_v)^e$$

ASSEMBLE $e = 1, 2, \ldots, n_{el}$

$$\boldsymbol{N}(\boldsymbol{U}) = \mathop{\mathsf{A}}_{e=1}^{n_{el}} (\boldsymbol{c}_v)^e$$

$$\boldsymbol{M}\dot{\boldsymbol{U}} = \mathop{\mathsf{A}}_{e=1}^{n_{el}} (\boldsymbol{m}_v)^e$$

$$\boldsymbol{K}\boldsymbol{U} = \mathop{\mathsf{A}}_{e=1}^{n_{el}} (\boldsymbol{k}_v)^e$$

$$\boldsymbol{G}\boldsymbol{P} = \mathop{\mathsf{A}}_{e=1}^{n_{el}} (\boldsymbol{g}_v)^e$$

$$\boldsymbol{G}^T\boldsymbol{U} = \mathop{\mathsf{A}}_{e=1}^{n_{el}} (\boldsymbol{g}_v^T)^e$$

ELEMENT-LEVEL MATRICES

$\underset{\sim}{m}$:

$$\underset{\sim}{w} \cdot \rho \frac{\partial \underset{\sim}{u}}{\partial t}$$

$$c_a^i N_a \underset{\sim}{e}^i \cdot \rho \; \dot{U}_b^j N_b \underset{\sim}{e}^j$$

$$c_a^i \quad \boxed{\rho \delta_{ij} \; N_a N_b} \quad \dot{U}_b^j$$

$\underset{\sim}{c}$:

$$\underset{\sim}{w} \cdot \rho \left(\underset{\sim}{u} \cdot \underset{\sim}{\nabla} \right) \Delta \underset{\sim}{u}$$

"FROZEN"

$$c_a^i \; N_a \underset{\sim}{e}^i \cdot \rho \left(\underset{\sim}{u} \cdot \underset{\sim}{\nabla} \right) \Delta U_b^j \; N_b \underset{\sim}{e}^j$$

$$c_a^i \quad \boxed{\rho \delta_{ij} N_a \left(\underset{\sim}{u} \cdot \underset{\sim}{\nabla} \right) N_b} \quad \Delta U_b^j$$

$\underset{\sim}{k}$:

$$\underset{\sim}{\varepsilon}(\underset{\sim}{w}) : 2\mu\, \underset{\sim}{\varepsilon}(\underset{\sim}{u})$$

$$\mu\, \underset{\sim}{\nabla}\underset{\sim}{w} : \left(\underset{\sim}{\nabla}\underset{\sim}{u} + (\underset{\sim}{\nabla}\underset{\sim}{u})^T \right)$$

$$\mu\left(\underset{\sim}{\nabla}\underset{\sim}{w} : \underset{\sim}{\nabla}\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{w} \cdot\cdot\, \underset{\sim}{\nabla}\underset{\sim}{u} \right)$$

$$\mu\Big(\underset{\sim}{\nabla}(c_a^i N_a \underset{\sim}{e}^i) : \underset{\sim}{\nabla}(U_b^j N_b \underset{\sim}{e}^j)$$

$$+ \underset{\sim}{\nabla}(c_a^i N_a \underset{\sim}{e}^i) \cdot\cdot\, \underset{\sim}{\nabla}(U_b^j N_b \underset{\sim}{e}^j) \Big)$$

$$c_a^i \mu \left(\underset{\sim}{\nabla} N_a\, \underset{\sim}{e}^i : \underset{\sim}{\nabla} N_b \underset{\sim}{e}^j + \underset{\sim}{\nabla} N_a \underset{\sim}{e}^i \cdot\cdot\, \underset{\sim}{\nabla} N_b \underset{\sim}{e}^j \right) U_b^j$$

$$c_a^i \quad \boxed{\mu\left(\delta_{ij}\, \underset{\sim}{\nabla} N_a \cdot \underset{\sim}{\nabla} N_b + \frac{\partial N_a}{\partial x_j} \frac{\partial N_b}{\partial x_i} \right)} \quad U_b^j$$

$\underset{\sim}{g}$:

$$(\underset{\sim}{\nabla}\cdot\underset{\sim}{w})\, p$$

$$\underset{\sim}{\nabla}\cdot(c_a^i N_a \underset{\sim}{e}^i) \quad P_b N_b$$

$$c_a^i \quad \underset{\sim}{e}^i\cdot\underset{\sim}{\nabla}N_a\, N_b \qquad P_b$$

$$c_a^i \qquad \boxed{\frac{\partial N_a}{\partial x_i}\; N_b} \qquad P_b$$

$\underset{\sim}{g}^T$:

$$q\,(\underset{\sim}{\nabla}\cdot\underset{\sim}{u})$$

$$c_a N_a \;\; \underset{\sim}{\nabla}\cdot(U_b^j N_b \underset{\sim}{e}^j)$$

$$c_a \qquad N_a\, \underset{\sim}{e}^j\cdot\underset{\sim}{\nabla}N_b \qquad U_b^j$$

$$c_a \qquad \boxed{N_a\;\frac{\partial N_b}{\partial x_j}} \qquad U_b^j$$

WE NEED TO COMPUTE ONE MORE ELEMENT-LEVEL MATRIX ...

$\underset{\sim}{C}$ IS ONLY ONE OF THE TWO MATRICES THAT

MAKE UP $\dfrac{\partial \underset{\sim}{N}}{\partial \underset{\sim}{U}}$

WE COMPUTE THE OTHER ONE AS FOLLOW ...

THE EXPANSION

$$\underset{\sim}{N}\left(\underset{\sim}{U}+\Delta \underset{\sim}{U}\right) = \underset{\sim}{N}(\underset{\sim}{U}) + \frac{\partial \underset{\sim}{N}}{\partial \underset{\sim}{U}} \Delta \underset{\sim}{U} + \dots$$

IS RELATED TO THE EXPANSION

$$\rho\Big(\left(\underset{\sim}{u}+\Delta \underset{\sim}{u}\right)\cdot\nabla\Big)\left(\underset{\sim}{u}+\Delta\underset{\sim}{u}\right) = \overbrace{\rho\left(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\right)\underset{\sim}{u}}^{\text{FROM THIS WE GET } \underset{\sim}{C}_V}$$

$$+ \underbrace{\rho\left(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\right)\Delta\underset{\sim}{u}}_{\text{FROM THIS WE GET } \underset{\sim}{C}} + \underbrace{\rho\left(\Delta\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\right)\underset{\sim}{u}}_{\substack{\text{FROM THIS WE GET}\\ \text{THE OTHER MATRIX}\\ \text{WE ARE LOOKING FOR}\\ \text{WE WILL CALL IT } \underset{\sim}{C}^{+}}} + \dots$$

HERE IS HOW WE CALCULATE $\underset{\sim}{C}^{+}$ ...

$\underset{\sim}{c}^{+}$:

$$\underset{\sim}{w} \cdot \rho (\Delta \underset{\sim}{u} \cdot \underset{\sim}{\nabla}) \underset{\sim}{u} \longleftarrow \text{"FROZEN"}$$

$$c_a^i \, N_a \, \underset{\sim}{e}^i \cdot \rho \left(\Delta U_b^j \, N_b \, \underset{\sim}{e}^j \cdot \underset{\sim}{\nabla} \right) \underset{\sim}{u}$$

$$c_\alpha^i \quad \rho \, N_\alpha N_b \, \underset{\sim}{e}^i \cdot (\underset{\sim}{e}^j \cdot \underset{\sim}{\nabla}) \underset{\sim}{u} \qquad \Delta U_b^j$$

$$c_\alpha^i \quad \boxed{\rho \; N_\alpha N_b \quad \frac{\partial u_i}{\partial x_j}} \qquad \Delta U_b^j$$

ASSEMBLY :

$$\frac{\partial \underset{\sim}{N}}{\partial \underset{\sim}{U}} = \overset{nel}{\underset{e=1}{\mathrm{A}}} \left((\underset{\sim}{c})^e + (\underset{\sim}{c}^{+})^e \right) \overset{\text{def}}{=} \underset{\sim}{C}$$

$$\underset{\sim}{M} = \overset{nel}{\underset{e=1}{\mathrm{A}}} (\underset{\sim}{m})^e$$

$$\underset{\sim}{K} = \overset{nel}{\underset{e=1}{\mathrm{A}}} (\underset{\sim}{k})^e$$

$$\underset{\sim}{G} = \overset{nel}{\underset{e=1}{\mathrm{A}}} (\underset{\sim}{g})^e$$

$$\underset{\sim}{G}^T = \overset{nel}{\underset{e=1}{\mathrm{A}}} (\underset{\sim}{g}^T)^e$$