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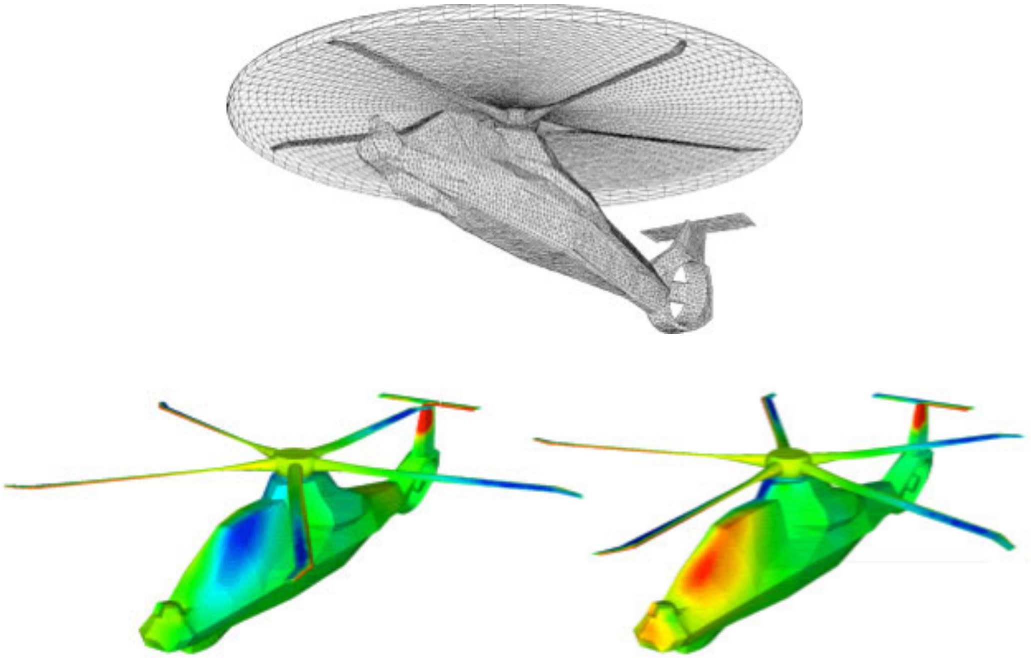

Time-Integration and Related Solution Techniques

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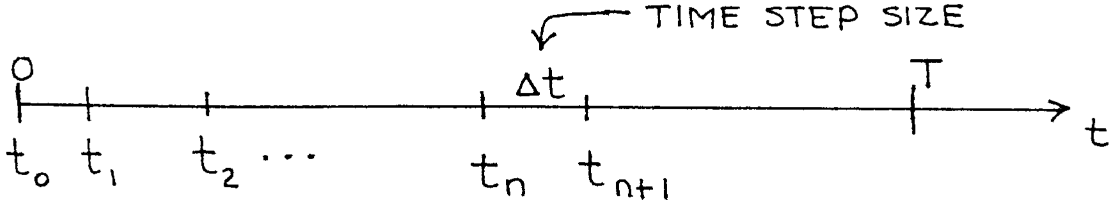
TIME-INTEGRATION

CONSIDER THE FOLLOWING LINEAR, ORDINARY DIFFERENTIAL EQUATION SYSTEM:

$$\underset{\sim}{M}\dot{\underset{\sim}{d}} + \underset{\sim}{K}\underset{\sim}{d} = \underset{\sim}{F}$$

$$\underset{\sim}{d}(0) = \underset{\sim}{d}_0$$

TIME-DISCRETIZATION:



n_{ts} : TOTAL NUMBER OF TIME STEPS

$$\underset{\sim}{d}_n \doteq \underset{\sim}{d}(t_n)$$

$\Delta t = t_{n+1} - t_n$: CAN BE VARIABLE

GIVEN: $\underset{\sim}{d}_n$, HOW DO WE CALCULATE $\underset{\sim}{d}_{n+1}$

THERE ARE MANY WAYS. WE DESCRIBE 3 OF THEM...

$$1) \quad \underset{\sim}{M}\left(\frac{\underset{\sim}{d}_{n+1}-\underset{\sim}{d}_{n}}{\Delta t}\right)+\underset{\sim}{K}\left(\underset{\sim}{d}_{n}\right)=\underset{\sim}{F}_{n}$$

(FORWARD DIFFERENCES)
(FORWARD EULER)

$$\downarrow$$

$$2) \qquad +\underset{\sim}{K}\left(\underset{\sim}{d}_{n+1}\right)=\underset{\sim}{F}_{n+1}$$

(BACKWARD DIFFERENCES)
(BACKWARD EULER)

$$\downarrow$$

$$3) \qquad +\underset{\sim}{K}\left(\frac{\underset{\sim}{d}_{n+1}+\underset{\sim}{d}_{n}}{2}\right)=\frac{1}{2}\left(\underset{\sim}{F}_{n+1}+\underset{\sim}{F}_{n}\right)$$

(CENTRAL DIFFERENCES)
(TRAPEZOIDAL RULE)

IN 1), IF WE APPROXIMATE $\underset{\sim}{M}$ WITH A DIAGONAL MATRIX, THEN TIME-MARCHING CAN BE DONE WITHOUT SOLVING A MATRIX SYSTEM.

A POSSIBLE APPROXIMATION TO $\underset{\sim}{M}$ IS $\underset{\sim}{M}_L$

$$\underset{\sim}{M}_L = \text{LUMP}(\underset{\sim}{M}) = \begin{array}{c}\text{ROW-SUM}\\ \text{OR}\\ \text{COLUMN-SUM}\end{array} = \begin{bmatrix} \int N_1\,d\Omega & & & \\ & \int N_2\,d\Omega & & \\ & & \ddots & \\ & & & \int N_N\,d\Omega \end{bmatrix}$$

DEFINITION:

EXPLICIT METHOD

A TIME-MARCHING METHOD THAT DOES NOT REQUIRE SOLUTION OF A MATRIX SYSTEM.

IMPLICIT METHOD

A TIME-MARCHING METHOD THAT REQUIRES SOLUTION OF A MATRIX SYSTEM.

NOTE THAT 2) AND 3) ARE IMPLICIT METHODS, AND, WITH $\underset{\sim}{M} \longleftarrow \underset{\sim}{M}_L$, 1) IS AN EXPLICIT METHOD.

GENERALIZED TRAPEZOIDAL FAMILY: $(0 \leqslant \alpha \leqslant 1)$

$$\underset{\sim}{M}\left(\frac{\underset{\sim}{d}_{n+1}-\underset{\sim}{d}_n}{\Delta t}\right)+\underset{\sim}{K}\underbrace{\left(\alpha \underset{\sim}{d}_{n+1}+(1-\alpha)\underset{\sim}{d}_n\right)}_{\overset{\text{def}}{=}\ \underset{\sim}{d}_{n+\alpha}} = \underbrace{\alpha \underset{\sim}{F}_{n+1}+(1-\alpha)\underset{\sim}{F}_n}_{\overset{\text{def}}{=}\ \underset{\sim}{F}_{n+\alpha}}$$

$$\Rightarrow \quad \underset{\sim}{M}\left(\frac{\underset{\sim}{d}_{n+1}-\underset{\sim}{d}_n}{\Delta t}\right)+\underset{\sim}{K}\,\underset{\sim}{d}_{n+\alpha} = \underset{\sim}{F}_{n+\alpha}$$

$\alpha = 0 \Longrightarrow 1)$ $(\underset{\sim}{M} \leftarrow \underset{\sim}{M}_L)$ $\qquad \alpha = 1 \Rightarrow 2)$ $\qquad \alpha = 1/2 \Longrightarrow 3)$

RE-ARRANGE THE EQUATION TO COLLECT THE UNKNOWN TERMS ON THE LEFT-HAND-SIDE:

$$\underbrace{(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K})}_{} \underset{\sim}{d}_{n+1} = \underbrace{(\underset{\sim}{M} - (1-\alpha) \Delta t \underset{\sim}{K})}_{} \underset{\sim}{d}_{n} + \Delta t \underset{\sim}{F}_{n+\alpha}$$

NEEDS TO BE
FORMED AND
FACTORIZED
IF $\alpha > 0$

↕

IMPLICIT METHOD

THESE MATRICES
DO NOT NEED
TO BE FORMED.
BECAUSE

$$\underset{\sim}{M} \underset{\sim}{d}_n = \mathop{\mathrm{A}}_{e=1}^{nel} (\underset{\sim}{m}_v)_n^e$$

$$\underset{\sim}{K} \underset{\sim}{d}_n = \mathop{\mathrm{A}}_{e=1}^{nel} (\underset{\sim}{k}_v)_n^e$$

IF Δt IS <u>SMALL ENOUGH</u>, THEN THERE COULD BE A JUSTIFICATION TO APPROXIMATE $(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K})$ by $\underset{\sim}{M}_L$.

THEN, THERE IS NO MATRIX FORMATION OR FACTORIZATION.

$\Longrightarrow$ <u>EXPLICIT METHOD</u> $\Longrightarrow$ ECONOMICAL

STABILITY : CONDITIONAL
(WILL CONVERGE IF $\Delta t < (\Delta t)_{\text{CRITICAL}}$)

ACCURACY : 1 ST-ORDER

(IF YOU HALVE Δt, THE ERROR WILL BE HALVED)

(<u>NOTE</u>:
IN A 2ND-ORDER METHOD: HALVE $\Delta t \Longrightarrow$ ERROR HALVED TWICE)

PROPERTIES OF THE GENERALIZED TRAPEZOIDAL FAMILY

$\alpha =$	0	1	1/2
NAME OF METHOD	FORWARD DIFF FORWARD EULER	BACKWARD DIFF BACKWARD EULER	CENTRAL DIFF TRAPEZOIDAL RULE
COST OF MATRIX FORMATION & FACTORIZATION	NO	YES	YES
IMPLICIT OR EXPLICIT	EXPLICIT	IMPLICIT	IMPLICIT
STABILITY	CONDITIONAL	UNCONDITIONAL	UNCONDITIONAL
ACCURACY	1 ST ORDER	1 ST ORDER	2 ND ORDER

PREDICTOR / MULTI-CORRECTOR METHOD

$$\left(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K}\right) \underbrace{\overset{\text{UNKNOWN}}{\underset{\sim}{d}_{n+1}}}_{\underset{\substack{\uparrow \\ \text{INITIAL GUESS} \\ \text{(PREDICTION)}}}{\underset{\sim}{d}^{(0)}_{n+1}} \; + \; \underset{\substack{\uparrow \\ \text{UNKNOWN INCREMENT} \\ \text{(CORRECTION)}}}{\Delta \underset{\sim}{d}^{(0)}_{n+1}}} = \left(\underset{\sim}{M} - (1-\alpha)\Delta t \underset{\sim}{K}\right) \underset{\sim}{d}_n + \Delta t \underset{\sim}{F}_{n+\alpha}$$

START WITH AN INITIAL GUESS : $\underset{\sim}{d}^{(0)}_{n+1} \leftarrow$ ZEROTH ITERATION

GIVEN: $\underset{\sim}{d}^{(i)}_{n+1}$ FIND: $\Delta \underset{\sim}{d}^{(i)}_{n+1}$

FROM:

$$\left(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K}\right) \underset{\substack{\uparrow \\ \text{CALCULATE THE} \\ \text{CORRECTION, AND} \\ \text{UPDATE:}}}{\Delta \underset{\sim}{d}^{(i)}_{n+1}} = \underbrace{\left(\underset{\sim}{M} - (1-\alpha)\Delta t \underset{\sim}{K}\right) \underset{\sim}{d}_n + \Delta t \underset{\sim}{F}_{n+\alpha} - \left(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K}\right) \underset{\sim}{d}^{(i)}_{n+1}}_{\substack{\text{WHEN THIS RESIDUAL} \\ \text{BECOMES SUFFICIENTLY} \\ \text{CLOSE TO ZERO,} \\ \text{OUR ITERATIONS} \\ \text{ARE ASSUMED TO} \\ \text{HAVE CONVERGED}}}$$

$$\underset{\sim}{d}^{(i+1)}_{n+1} = \underset{\sim}{d}^{(i)}_{n+1} + \Delta \underset{\sim}{d}^{(i)}_{n+1}$$

REMARKS:

1. IF $(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K})$ IS LEFT AS IT IS, THEN IT TAKES ONLY <u>ONE ITERATION</u> TO CONVERGE.

2. WE CAN APPROXIMATE $(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K})$ ANYWAY WE WISH, PROVIDED THAT THE ITERATIONS CONVERGE.

3. THE OBJECTIVE IS TO APPROXIMATE $(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K})$ WITH SOMETHING SUFFICIENTLY SIMPLE, BUT NOT TOO SIMPLE TO CONVERGE WITHIN REASONABLE NUMBER OF ITERATIONS.

4. EXAMPLE: $(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K}) \leftarrow \underset{\sim}{M}_L$

   - LIMIT ON Δt
   - CONVERGES SLOWLY

5. EXAMPLES: $(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K}) \leftarrow (\underset{\sim}{M}_L + \alpha \Delta t \, \text{DIAG}(\underset{\sim}{K}))$

   $(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K}) \leftarrow \text{DIAG}(\underset{\sim}{M} + \alpha \Delta t \underset{\sim}{K})$

SOLUTION TECHNIQUES FOR NONLINEAR EQUATIONS

CONSIDER THE FOLLOWING SYSTEM OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS IN TIME :

$$\underset{\sim}{M}\,\dot{\underset{\sim}{U}} + \underset{\sim}{N}(\underset{\sim}{U}) + \underset{\sim}{K}\,\underset{\sim}{U} = \underset{\sim}{F}$$

$$\underset{\sim}{U}(0) = \underset{\sim}{U}_0$$

THIS CAN BE DISCRETIZED AS FOLLOWS :

$$\underbrace{\underset{\sim}{M}\left(\frac{\underset{\sim}{U}_{n+1}-\underset{\sim}{U}_n}{\Delta t}\right)}_{\boxed{\text{LINEAR TERMS}}} + \underbrace{\underset{\sim}{N}\left(\alpha\,\underset{\sim}{U}_{n+1}+(1-\alpha)\,\underset{\sim}{U}_n\right)}_{\boxed{\text{NONLINEAR VECTOR FUNCTION}}}$$

$$+ \overbrace{\underset{\sim}{K}\left(\alpha\,\underset{\sim}{U}_{n+1}+(1-\alpha)\,\underset{\sim}{U}_n\right)} = \underset{\sim}{F}_{n+\alpha}$$

THIS MEANS THAT IN NONLINEAR PROBLEMS,

INSTEAD OF A LINEAR EQUATION SYSTEM

$$(\underset{\sim}{M} + \alpha\Delta t\,\underset{\sim}{K})\,\underset{\sim}{d}_{n+1} = \cdots\cdots ,$$

WE NEED TO SOLVE AT EVERY TIME STEP

A NONLINEAR EQUATION SYSTEM $\underset{\sim}{N}(\underset{\sim}{d}_{n+1}) = \cdots\cdots$

FOR NOTATIONAL CONVENIENCE, DROP THE SUBSCRIP $n+1$ AND CALL THE R.H.S. $\underset{\sim}{F}$. THEN,

$$\boxed{\underset{\sim}{N}(\underset{\sim}{d}) = \underset{\sim}{F}}$$

IS WHAT NEEDS TO BE SOLVED AT EVERY TIME STEP.

NEWTON-RAPHSON METHOD

IN SOLVING

$$\underset{\sim}{N}(\underset{\uparrow\ \text{UNKNOWN}}{\underset{\sim}{d}}) = \underset{\sim}{F} \quad ,$$

START WITH AN INITIAL GUESS $\underset{\sim}{d}^{0}$, ← ZEROTH ITERATION

AND AT EACH ITERATION, GIVEN $\underset{\sim}{d}^{i}$, CALCULATE A CORRECTION $\Delta\underset{\sim}{d}^{i}$, SUCH THAT

$$\underset{\sim}{N}(\underset{\sim}{d}^{i} + \Delta\underset{\sim}{d}^{i}) \doteq \underset{\sim}{F} \ .$$

TO CALCULATE $\Delta\underset{\sim}{d}^{i}$, CONSIDER THE FOLLOWING TRUNCATED TAYLOR SERIES EXPANSION AROUND $\underset{\sim}{d}^{i}$:

$$\underset{\sim}{N}(\underset{\sim}{d}^{i} + \Delta\underset{\sim}{d}^{i}) = \underset{\sim}{N}(\underset{\sim}{d}^{i}) + \left.\frac{\partial\underset{\sim}{N}}{\partial\underset{\sim}{d}}\right|_{\underset{\sim}{d}^{i}} \Delta\underset{\sim}{d}^{i} .$$

THEN SOLVE

$$\underbrace{\left.\frac{\partial\underset{\sim}{N}}{\partial\underset{\sim}{d}}\right|_{\underset{\sim}{d}^{i}}}_{\underset{\sim}{K}_{T}(\underset{\sim}{d}^{i})\ :\ \text{"TANGENT STIFFNESS" MATRIX}} \Delta\underset{\sim}{d}^{i} = \underset{\sim}{F} - \underset{\sim}{N}(\underset{\sim}{d}^{i}) \ ,$$

AND UPDATE :

$$\underset{\sim}{d}^{i+1} = \underset{\sim}{d}^{i} + \Delta\underset{\sim}{d}^{i}$$

NEWTON-RAPHSON METHOD CONVERGES QUADRATICALLY, PROVIDED THAT

1) THE INITIAL GUESS $\underset{\sim}{d}^{0}$ IS SUFFICIENTLY CLOSE TO THE SOLUTION, AND

2) INVERSE OF $\dfrac{\partial \underset{\sim}{N}}{\partial \underset{\sim}{d}}$ EXISTS.

HOW DO WE DECIDE WHEN TO STOP ITERATIONS?

ANSWER: WHEN

a) $\underset{\sim}{F}-\underset{\sim}{N}(\underset{\sim}{d}^{i})$ IS SUFFICIENTLY CLOSE TO $\underset{\sim}{0}$, OR

b) A PREDETERMINED NUMBER OF ITERATIONS HAVE BEEN PERFORMED.

IN BOTH CASES, THE RESIDUAL $\underset{\sim}{F}-\underset{\sim}{N}(\underset{\sim}{d}^{i})$ NEED TO BE MONITORED. THE RESIDUAL AT ITERATION i:

$$\underset{\sim}{R}^{i} = \underset{\sim}{F}-\underset{\sim}{N}(\underset{\sim}{d}^{i}).$$

CHECK TO SEE IF

$$\|\underset{\sim}{R}^{i}\| \leqslant \varepsilon \longleftarrow \text{A PREDETERMINED, SMALL VALUE}$$

EXAMPLE FOR THE NORM $\|\cdot\|$:

$$\|\underset{\sim}{x}\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}.$$

TO CHECK CONVERGENCE IN A WAY THAT MAKES SENSE, WE NEED TO SCALE $\|\underset{\sim}{R}^{i}\|$. ONE WAY IS TO SCALE IT WITH $\|\underset{\underset{\text{INITIAL RESIDUAL}}{\uparrow}}{\underset{\sim}{R}}{}^{0}\|$.

THEREFORE, THE CONVERGENCE CHECK TAKES THE FORM :

$$\frac{\|\underset{\sim}{R}^{i}\|}{\|\underset{\sim}{R}^{0}\|} \overset{?}{\leqslant} \varepsilon$$

QUITE OFTEN, IT IS MORE PRACTICAL, ESPECIALLY IN SOLVING TIME-DEPENDENT PROBLEMS, TO PERFORM A PREDETERMINED NUMBER OF ITERATIONS AT EACH TIME STEP, BUT AT THE SAME TIME KEEP AN EYE ON

$$\frac{\|\underset{\sim}{R}^{i}\|}{\|\underset{\sim}{R}^{0}\|} ,$$

TO MAKE SURE THAT IT DOES NOT EXCEED A CERTAIN LEVEL.

MODIFIED NEWTON-RAPHSON METHOD

$$\underbrace{\underset{\sim}{K}_T(\underset{\sim}{d}^i)}_{\uparrow}\,\Delta\underset{\sim}{d}^i = \underset{\sim}{F} - \underset{\sim}{N}(\underset{\sim}{d}^i)$$

IN THE "FULL" NEWTON-RAPHSON METHOD

THIS GETS UPDATED AT EVERY ITERATION

BY $\underset{\sim}{K}_T(\underset{\sim}{d}^i) = \left.\dfrac{\partial \underset{\sim}{N}}{\partial \underset{\sim}{d}}\right|_{\underset{\sim}{d}^i}$

IN THE <u>MODIFIED</u> NEWTON-RAPHSON METHOD

$\underset{\sim}{K}_T$ IS <u>NOT</u> UPDATED EVERY ITERATION.

AT SOME OF THE ITERATIONS, OLD VALUES

OF $\underset{\sim}{K}_T$ ARE USED.

THIS APPROACH

- CUTS COST OF COMPUTATION
- BUT MAY REDUCE THE CONVERGENCE RATE.

INCREMENTAL NEWTON-RAPHSON METHOD

THE NEWTON-RAPHSON METHOD MAY HAVE CONVERGENCE DIFFICULTIES IF THE INITIAL GUESS $\underset{\sim}{d}^{0}$ IS TOO "FAR" FROM THE SOLUTION.

ONE APPROACH : INCREMANTAL "LOADING" (i.e. "LOAD" RAMPING)

$$\underset{\sim}{N}(\underset{\sim}{d}) = \underset{\sim}{F}$$

1) $\uparrow$ START WITH A FRACTION OF THIS. THIS MEANS A "LOAD" SMALLER THAN THE ACTUAL ONE.

2) GET A CONVERGED SOLUTION FOR THIS LOAD LEVEL

3) USE THAT SOLUTION AS THE INITIAL GUESS FOR THE NEXT LEVEL.

4) REPEAT 2) AND 3) UNTIL THE FULL LOAD LEVEL IS REACHED.

EXAMPLE : START WITH $\frac{1}{10}\underset{\sim}{F} \xrightarrow{\text{GET}} \underset{\sim}{d}_{1/10}$

USE $\underset{\sim}{d}_{1/10}$ AS INITIAL GUESS FOR $\frac{2}{10}\underset{\sim}{F} \xrightarrow{\text{GET}} \underset{\sim}{d}_{2/10}$

USE $\underset{\sim}{d}_{2/10}$ " " " " $\frac{3}{10}\underset{\sim}{F} \xrightarrow{\text{GET}} \underset{\sim}{d}_{3/10}$

⋮

USE $\underset{\sim}{d}_{9/10}$ " " " " $\underset{\sim}{F} \xrightarrow{\text{GET}} \underset{\sim}{d}$

RAMPING UP THE REYNOLDS NUMBER

IN FLUID DYNAMICS PROBLEMS, TYPICALLY, THE NONLINEAR TERMS ARE PROPORTIONAL TO THE REYNOLDS NUMBER.

ONE CAN START WITH A FRACTION OF THE ACTUAL REYNOLDS NUMBER,

2) GET A CONVERGED SOLUTION FOR THAT REYNOLDS NUMBER

3) USE THAT SOLUTION AS THE INITIAL GUESS FOR THE NEXT LEVEL OF THE REYNOLDS NUMBER

4) REPEAT 2) AND 3) UNTIL THE FULL REYNOLDS NUMBER IS REACHED.

EXAMPLE : START WITH $Re \times 10^{-5} \xrightarrow{GET} \underset{\sim}{d}_{\times 10^{-5}}$

USE $\underset{\sim}{d}_{\times 10^{-5}}$ AS INITIAL GUESS FOR $Re \times 10^{-4} \xrightarrow{GET} \underset{\sim}{d}_{\times 10^{-4}}$

USE $\underset{\sim}{d}_{\times 10^{-4}}$ " " " " $Re \times 10^{-3} \xrightarrow{GET} \underset{\sim}{d}_{\times 10^{-3}}$

⋮

USE $\underset{\sim}{d}_{\times 10^{-1}}$ " " " " $Re \xrightarrow{GET} \underset{\sim}{d}$.

NOTE : IN A TIME-DEPENDENT PROBLEM, RAMPING THE REYNOLDS NUMBER NEEDS TO BE EXERCISED ONLY IN THE INITIAL STAGES OF THE TIME-MARCHING.

ITERATIVE SOLUTION TECQNIQUES FOR LINEAR EQUATION SYSTEMS

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

WHERE DO THESE SYSTEMS COME FROM ?

EXAMPLE :

NONLINEAR PROBLEMS }
$$\underbrace{\underset{\sim}{K}_T(\underset{\sim}{d}^i)}_{\underset{\sim}{A}}\;\underbrace{\Delta\underset{\sim}{d}^i}_{\underset{\sim}{x}} = \underbrace{\underset{\sim}{F}-\underset{\sim}{N}(\underset{\sim}{d}^i)}_{\underset{\sim}{b}}$$

LINEAR PROBLEMS }
$$\overbrace{(\underset{\sim}{M}+\alpha\Delta t\underset{\sim}{K})}\;\overbrace{\Delta\underset{\sim}{d}^{(i)}_{n+1}} = \overbrace{\cdots\cdots}$$

IN MOST CASES, ESPECIALLY THOSE IN 3D, THE MATRIX $\underset{\sim}{A}$ IS TOO LARGE, AND THEREFORE WE CANNOT USE A "DIRECT" SOLUTION METHOD SUCH AS THE GAUSSIAN ELIMINATION METHOD OR OTHER FACTORIZATION TECQNIQUES. BECASE WE CANNOT AFFORD TO

a) FACTORIZE $\underset{\sim}{A}$

b) STORE THE FACTORIZED $\underset{\sim}{A}$.

IN ITERATIVE SOLUTION OF

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b} \;,$$

$$\uparrow \text{ UNKNOWN}$$

START WITH AN INITIAL GUESS $\underset{\sim}{x}_0 \leftarrow$ ZEROTH ITERATION AND AT EACH ITERATION, GIVEN $\underset{\sim}{x}_m$, CALCULATE A CORRECTION $\Delta \underset{\sim}{x}_m$, SUCH THAT

$$\underset{\sim}{A}\,(\underset{\sim}{x}_m + \Delta \underset{\sim}{x}_m) = \underset{\sim}{b}$$

OR

$$\underset{\sim}{A}\,\Delta \underset{\sim}{x}_m = \underset{\sim}{b} - \underset{\sim}{A}\,\underset{\sim}{x}_m$$

AND UPDATE

$$\underset{\sim}{x}_{m+1} = \underset{\sim}{x}_m + \Delta \underline{x}_m \;.$$

- IF WE LEAVE $\underset{\sim}{A}$ AS IT IS, THIS PROCEDURE IS ESSENTIALLY NO DIFFERENT THAN SOLVING $\underset{\sim}{A}\,\underline{x} = \underset{\sim}{b}$, AND IT CONVERGES IN ONE ITERATION TO THE SOLUTION OF $\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$. FOR EXAMPLE, IF WE START WITH $\underset{\sim}{x}_0 = \underset{\sim}{0}$, THEN $\Delta \underset{\sim}{x}_1 = \underset{\sim}{x}$. BUT ....

IN MOST CASES, WE CANNOT AFFORD LEAVING $\underset{\sim}{A}$ AS IT IS. THEREFORE WE APPROXIMATE $\underset{\sim}{A}$ WITH A "PRECONDITIONING" MATRIX $\underset{\sim}{P}$, AND SOLVE AT EVERY ITERATION

$$\underset{\sim}{P}\,\Delta \underset{\sim}{y}_m = \underset{\sim}{b} - \underset{\sim}{A}\,\underset{\sim}{x}_m$$

AND UPDATE $\underset{\sim}{x}_m$ BY USING AN UPDATE METHOD MORE SOPHISTICATED THAN SIMPLY ADDING $\Delta \underset{\sim}{y}_m$ to $\underset{\sim}{x}_m$.

THERE ARE 3 MAIN ISSUES HERE.

1) HOW TO COMPUTE $\underset{\sim}{b} - \underset{\sim}{A}\,\underset{\sim}{x}_m$ IN THE MOST EFFICIENT WAY.

2) HOW TO DESIGN $\underset{\sim}{P}$ IN A WAY THAT KEEPS IT SIMPLE WITHOUT SLOWING THE CONVERGENCE TOO MUCH.

3) HOW TO UPDATE $\underset{\sim}{x}_m$ IN A WAY THAT IS ECONOMICAL BUT ALSO HELPS CONVERGENCE.

1) HOW TO COMPUTE $\underset{\sim}{b} - \underset{\sim}{A}\,\underset{\sim}{x}_m$

a) SPARSE-MATRIX-BASED :

STORE $\underset{\sim}{A}$ BY USING A SPARSE-MATRIX STORAGE TECHNIQUE, AND PERFORM THE COMPUTATION WITH GLOBAL MATRIX-VECTOR PRODUCTS.

b) ELEMENT-MATRIX-BASED :

NOTE THAT: $$\underset{\sim}{A}\,\underset{\sim}{x} = \underbrace{\Big(\mathop{\mathsf{A}}_{e=1}^{nel} \underset{\sim}{A}^{e}\Big)}_{\text{BUT WE DO NOT NEED THIS}} \underset{\sim}{x}$$

BECAUSE

$$= \mathop{\mathsf{A}}_{e=1}^{nel} \underbrace{\Big(\overset{\swarrow\ \text{ELEMENT MATRIX}}{\underset{\sim}{A}^{e}}\,\underset{\sim}{x}\Big)}_{\uparrow\ \text{ELEMENT VECTOR}}$$

c) ELEMENT-VECTOR-BASED :

$$\underset{\sim}{A}\,\underset{\sim}{x}_m = \frac{\partial \underset{\sim}{N}}{\partial \underset{\sim}{d}}\,\underset{\sim}{x}_m = \lim_{\varepsilon \to 0} \frac{\underset{\sim}{N}(\underset{\sim}{d} + \varepsilon \underset{\sim}{x}_m) - \underset{\sim}{N}(\underset{\sim}{d})}{\varepsilon}$$

$$\Rightarrow \quad \underset{\sim}{A}\,\underset{\sim}{x}_m \doteq \frac{\mathop{\mathsf{A}}_{e=1}^{nel} \underset{\sim}{N}^{e}(\underset{\sim}{d} + \varepsilon \underset{\sim}{x}_m) - \mathop{\mathsf{A}}_{e=1}^{nel} \underset{\sim}{N}^{e}(\underset{\sim}{d})}{\varepsilon}$$

WHERE ε IS A SMALL NUMBER.

2) HOW TO DESIGN $\underset{\sim}{P}$

$\underset{\sim}{P} = \text{DIAG}(\underset{\sim}{A})$ IS THE SIMPLEST CHOICE.

↳ DIAGONAL OR NODAL-BLOCK-DIAGONAL

CONVERGES SLOWLY, ESPECIALLY FOR INCOMPRESSIBLE FLOWS.

OTHER POSSIBILITIES :
- CLUSTERED ELEMENT-BY-ELEMENT PRECONDITIONING
- MIXED PRECONDITIONING

3) HOW TO UPDATE $\underset{\sim}{x}_m$

AS A WAY JUST ONE STEP MORE SOPHISTICATED THAN SIMPLY ADDING $\Delta \underset{\sim}{y}_m$ to $\underset{\sim}{x}_m$,

WE UPDATE $\underset{\sim}{x}_m$ BY USING THE EXPRESSION

$$\underset{\sim}{x}_{m+1} = \underset{\sim}{x}_m + s\,\Delta \underset{\sim}{y}_m$$

↳ A SCALAR "SEARCH" PARAMETER

WE DETERMINE THE "BEST" "S" BY MINIMIZING THE UPDATED RESIDUAL $\underset{\sim}{r}_{m+1} = \underset{\sim}{b} - \underset{\sim}{A}\,\underset{\sim}{x}_{m+1}$ WITH RESPECT TO "S".

$$\underset{\sim}{r}_{m+1} = \underset{\sim}{b} - \underset{\sim}{A}\left(\underset{\sim}{x}_m + s\,\Delta \underset{\sim}{y}_m\right) = \overbrace{\underset{\sim}{b} - \underset{\sim}{A}\,\underset{\sim}{x}_m}^{\underset{\sim}{r}_m} - s\,\underset{\sim}{A}\,\Delta \underset{\sim}{y}_m$$

$$\underbrace{\underset{\sim}{r}_{m+1}}_{\text{MINIMIZE THIS}} = \underbrace{\underset{\sim}{r}_m}_{\text{KNOWN}} - \underset{\uparrow}{s}\ \underbrace{\underset{\sim}{A}\,\Delta\underset{\sim}{y}_m}_{\text{KNOWN}}$$

$\longrightarrow$ WITH RESPECT TO THIS

MINIMIZE $\|\underset{\sim}{r}_{m+1}\|$ WITH RESPECT TO "s". $\Longleftrightarrow$

MINIMIZE $\|\underset{\sim}{r}_{m+1}\|^2$ WITH RESPECT TO "s". $\Longleftrightarrow$

$$\frac{d}{ds}\|\underset{\sim}{r}_{m+1}\|^2 = 0 \quad \Longleftrightarrow \quad \frac{d}{ds}\left(\underset{\sim}{r}_{m+1}\cdot\underset{\sim}{r}_{m+1}\right) = 0$$

$$\Rightarrow \quad \underset{\sim}{r}_{m+1}\cdot\frac{d}{ds}\left(\underset{\sim}{r}_{m+1}\right) = 0$$

$$\Rightarrow \quad \underset{\sim}{r}_{m+1}\cdot\left(-\underset{\sim}{A}\,\Delta\underset{\sim}{y}_m\right) = 0$$

$$\Rightarrow \quad \left(\underset{\sim}{r}_m - s\underset{\sim}{A}\,\Delta\underset{\sim}{y}_m\right)\cdot\left(\underset{\sim}{A}\,\Delta\underset{\sim}{y}_m\right) = 0$$

$$\Rightarrow \quad s = \frac{\underset{\sim}{r}_m\cdot\left(\underset{\sim}{A}\,\Delta\underset{\sim}{y}_m\right)}{\left(\underset{\sim}{A}\Delta\underset{\sim}{y}_m\right)\cdot\left(\underset{\sim}{A}\,\Delta\underset{\sim}{y}_m\right)}$$

OR

$$= \frac{\underset{\sim}{r}_m\cdot\left(\underset{\sim}{A}\,\Delta\underset{\sim}{y}_m\right)}{\|\underset{\sim}{A}\,\Delta\underset{\sim}{y}_m\|^2}$$

THERE ARE MANY OTHER WAYS, INCLUDING METHODS WITH MULTIPLE "SEARCH" PARAMETERS AND DIRECTIONS.